4 LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT CO-EFFICIENTS

4.1. DEFINITION

A linear differential equation with constant co-efficients is the one in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together and the coefficients are constants.

A differential equation of the form

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots.. + P_n y = X \quad \ldots.(1)$$

where $P_0, P_1, P_2, \ldots.., P_n$ and X are functions of x or constants, is called a *linear differential equation of nth order.*

And if $P_0, P_1, P_2, \ldots.., P_n$ are all constants (not functions of x) and X is some function of x, then the equation is a *linear differential equation with constant co-efficients of nth order.*

4.2. THE DIFFERENTIAL OPERATOR 'D'

The part $\frac{d}{dx}$ of the symbol $\frac{dy}{dx}$ may be regarded as a differential operator, such that when it operates on y, the result is the derivative of y.

It is usual to write D for $\frac{d}{dx}$, D^2 for $\frac{d^2}{dx^2}$, D^3 for $\frac{d^3}{dx^3}$, ..., D^n for $\frac{d^n}{dx^n}$.

In terms of the operator D, the differential equation (1) can be written as

$$(P_0 D^n + P_1 D^{n-1} + P_2 D^{n-2} + \ldots.. + P^n)\, y = X$$

4.3. COMPLETE SOLUTION OF THE LINEAR DIFFERENTIAL EQUATION

Theorem. *If $y = y_1, y = y_2, ..., y = y_n$ are linear independent solutions of $(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n) y = 0$ then $y = c_1 y_1 + c_2 y_2 + ... + c_n y_n$ is the general or complete solution of the differential equation (1), where $c_1, c_2, c_3, ..., c_n$ are n arbitrary constants.*

Proof. The given equation is

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + ... + a_n y = 0 \quad ...(1)$$

Since $y = y_1, y = y_2, ..., y = y_n$ are solutions of equation (1), therefore $y_1, y_2, ..., y_n$ must satisfy (1).

$$\therefore \quad a_0 D^n y_1 + a_1 D^{n-1} y_1 + a_2 D^{n-2} y_1 + ... + a_n y_1 = 0$$

$$a_0 D^n y_2 + a_1 D^{n-1} y_2 + a_2 D^{n-2} y_2 + ... + a_n y_2 = 0$$

$$.. \quad ..... \quad ...(2)$$

$$.. \quad .....$$

$$a_0 D^n y_n + a_1 D^{n-1} y_n + a_2 D^{n-2} y_n + ... + a_n y_n = 0$$

Multiplying above equations successively by $c_1, c_2, ..., c_n$ and adding vertically, we get

$$a_0 D^n [c_1 y_1 + c_2 y_2 + ... + c_n y_n] + a_1 D^{n-1} [c_1 y_1 + c_2 y_2 + ... + c_n y_n]$$

$$+ a_n D^{n-2} [c_1 y_1 + c_2 y_2 + ... + c_n y_n] + ...$$

$$........... + a_n [c_1 y_1 + c_2 y_2 + ... + c_n y_n] = 0$$

This shows that $y = c_1 y_1 + c_2 y_2 + ... + c_n y_n$ is also a solution of equation (1). As the solution contains n arbitrary constants, so this *solution is the general solution or complete solution of the differential equation (1).*

4.4. AUXILIARY EQUATION (A.E.)

Definition. *The equation obtained by equating to zero the symbolic coefficient of y is called the auxiliary equation, provided D is taken as an algebraic quantity.*

Consider the differential equation

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a^n) y = 0 \quad ...(1)$$

where $a_0, a_1, a_2, ..., a_n$ are all constants.

Let $y = e^{mx}$ be a solution of this equation (1).

Then putting $y = e^{mx}$, $Dy = me^{mx}$, $D^2y = m^2e^{mx}$,......, $D^ny = m^n e^{mx}$ in equation (1), we have

$$(a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + ... + a_n) e^{mx} = 0$$

Cancelling e^{mx} ($\because e^{mx} \neq 0$ for any m), we have

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + ... + a_n = 0 \quad ...(2)$$

Hence e^{mx} will be root of equation (1) if m is a root of the algebraic equation (2).

Equation (2) is called the auxiliary equation for the differential equation (1).

Remark.

It is observed that the values of D obtained on solving the equation

$$a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n = 0$$

are same as the values of m obtained on solving equation (2).

Hence $a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n = 0$ can be regarded as the auxiliary equation in place of equation $a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + ... + a_n = 0$.

Therefore in practice we do not replace D by m to form the auxiliary equation. The equation in D may be regarded as auxiliary equation which is obtained by equating to zero the symbolic co-efficient of y in equation (1).

4.5. TO FIND THE COMPLETE SOLUTION OF THE DIFFERENTIAL EQUATION

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n) y = 0.$$

The given equation is

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n) y = 0. \quad ...(1)$$

The auxiliary equation for (1) is

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + ... + a_n = 0. \quad ...(2)$$

Now the following different cases arise :

Case I. *When all the roots of auxiliary equation* (2) *are real and different.*

Let $m_1, m_2, m_3, ..., m_n$ be the n different and real roots of equation (2).

Then $y = e^{m_1 x}$, $y = e^{m_2 x}$, ..., $y = e^{m_n x}$ are n independent solutions of (1).

1. Find the roots of the auxiliary equation

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots + a_n = 0$$

2. Find the complete solution as shown in the following table.

Roots of Auxiliary equation	Complete solution
Case I. All roots, $m_1, m_2, m_3, \ldots, m_n$ are real and different.	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots + c_n e^{m_n x}$.
Case II. $m_1 = m_2$ are two real roots and other roots are real and different.	$y = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \ldots + c_n e^{m_n x}$.
Case III. (Imaginary Roots) $\alpha \pm i\beta$, are two imaginary roots and other roots are real.	Corresponding part of the general solution $e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$ or $c_1 e^{\alpha x} \cos(\beta x + c_2)$ or $c_1 e^{\alpha x} \sin(\beta x + c_2)$
Case IV. $\alpha \pm i\beta$, $\alpha \pm i\beta$ are four imaginary roots and other roots are real.	Corresponding part of complete solution is $e^{\alpha x}[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x) \sin \beta x]$

SOLVED EXAMPLES

Example 1.

If the two roots of the auxiliary equation $\lambda^2 + a_1\lambda + a_2 = 0$ are real and equal, then prove that $y = (c_1 + c_2 x)\, e^{\lambda x}$ is a solution of the equation $y_2 + a_1 y_1 + a_2 y = 0$, where c_1, c_2 are arbitrary constants.

Solution. The given equation is

$$y_2 + a_1 y_1 + a_2 y = 0$$

Equation in symbolic form is

$$D^2 y + a_1 D y + a_2 y = 0$$

or

$$(D^2 + a_1 D + a_2)\, y = 0$$

Auxiliary equation (A.E.) is

$$\lambda^2 + a_1 \lambda + a_2 = 0$$

The roots of (2) are given to be real and equal.

Let each root $= \lambda$

$\therefore$ Equation (1) becomes

$$(D-\lambda)^2 y = 0$$

$$(D-\lambda)(D-\lambda)y = 0$$

Putting $(D-\lambda)y = v$, we get

$$(D-\lambda)v = 0$$

or $$\frac{dv}{dx} = \lambda v$$

or $$\frac{dv}{v} = \lambda\, dx$$

Integrating, $$\log v = \lambda x + \log c$$

or $$\log v - \log c = \lambda x$$

or $$\log \frac{v}{c} = \lambda x$$

or $$\frac{v}{c} = e^{\lambda x} \Rightarrow v = ce^{\lambda x}$$

$\therefore$ $$(D-\lambda)y = ce^{\lambda x}$$

or $$\frac{dy}{dx} - \lambda y = c.e^{\lambda x},$$

which is a linear differential equation of the first order.

$$\text{I.F.} = e^{\int -\lambda dx} = e^{-\lambda x}$$

Hence, the complete solution is

$$y \,.\, e^{-\lambda x} = \int c \,.\, e^{\lambda x} \,.\, e^{-\lambda x}\, dx + c'$$

$$= \int c\, dx + c' = cx + c'$$

or $$y = (cx + c')\, e^{\lambda x}$$

or $$y = (c_1 + c_2 x)\, e^{\lambda x}.$$

Let (1) $D = m_1, m_2, m_3 - m_n$ real and distinct.

$m_1 \neq m_2$.

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} - - - c_n e^{m_n x}$.

(2) Let $m_1, m_1 - - . m_n$

$m_1 = m_1$ real and distinct

$y = (c_1 + c_2 x) e^{m_1 x} - + - - c_n e^{m_n x}$.

(3) $D = \alpha \to$ real part

$\beta \to$ imaginary part

complete $\because v = (D-\lambda)y]$ Sol^n.

$y = e^{\alpha x}[c_1 \cos \beta x + c_2 \sin \beta x]$

$D = \alpha + i\beta, \ \alpha + i\beta$

Sol$^n \to y = [(c_1 + c_2) e^{\alpha x} \cos \beta x + (c_1 + c_2) \sin \beta x]$

(4) $y = e^{-\alpha x}[c_1 \cos \beta x + c_2 \sin \beta x] + e^{\alpha/2}[c_3 \cos \frac{\beta x}{\sqrt{2}} + c_4 \sin \beta x]$

Example 2.

If λ_1, λ_2 are real and distinct roots of the auxiliary equation $\lambda^2 + a_1\lambda + a_2 = 0$ and $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$ are its solutions, then prove that $y = c_1 y_1 + c_2 y_2$ is a solution of

$$\frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0$$

Is this solution general? [M.D.U. 2001]

Solution. The auxiliary equation is $\lambda^2 + a_1 \lambda + a_2 = 0$ ...(1)

Since its roots are λ_1 and λ_2 $(\lambda_1 \neq \lambda_2)$

$\therefore$ $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ are two independent solutions of

$$D^2 y + a_1 Dy + a_2 y = 0 \quad \ldots(2)$$

$\therefore$

$$D^2 y_1 + a_1 D y_1 + a_2 y_1 = 0 \quad \ldots(3)$$

and

$$D^2 y_2 + a_1 D y_2 + a_2 y_2 = 0 \quad \ldots(4)$$

Multiplying (3) by c_1, (4) by c_2 and adding, we get

$$D^2 [c_1 y_1 + c_2 y_2] + a_1 D [c_1 y_1 + c_2 y_2] + a_2 [c_1 y_1 + c_2 y_2] = 0$$

This shows that $y = c_1 y_1 + c_2 y_2$ is a solution of equation

$$D^2 y + a_1 Dy + a_2 y = 0$$

i.e., $y = c_1 y_1 + c_2 y_2$ is a solution of equation

$$\frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0$$

Since the solution involves two arbitrary constants and the order of the given equation is also 2, therefore this solution is general.

Example 3.

Solve the differential equation two Value of D is not same.

$y = c_1 e^{x} + c_2 e^{2x}$

$$2\frac{d^3 y}{dx^3} - 7\frac{d^2 y}{dx^2} + 7\frac{dy}{dx} - 2y = 0.$$

Solution. The given equation is

$$2\frac{d^3 y}{dx^3} - 7\frac{d^2 y}{dx^2} + 7\frac{dy}{dx} - 2y = 0$$

Equation in symbolic form is

$$(2D^3 - 7D^2 + 7D - 2)\, y = 0$$ Put

Auxiliary equation is $2D^3 - 7D^2 + 7D - 2 = 0^*$ $D = 1.$

$$\begin{array}{c|ccc} 1 & 2 & -7 & 7 \\ & & +2 & -5 \\ \hline & 2 & -5 & 2 \end{array}$$

i.e., $(D - 1)(D - 2)(2D - 1) = 0$

$\therefore$ $D = 1, 2, \frac{1}{2}$

$2D^2 - 5D + 2 = 0$

$2D^2 - 4D - D + 2 = 0$

$2(D - 2) - 1(D - 2) = 0$

$2D - 1 = 0$; $D = 2$

$D = 1/2$

Here, we have three real and different roots.

Hence, the complete solution is $y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{x/2}$.

* As we have already mentioned in Remark on page 4.3, we shall be taking the auxiliary equation in D instead of m throughout this book for the sake of convenience.

Example 4.

Solve the differential equation

$$\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} - 11\frac{dy}{dx} - 4y = 0.$$

By using Synthesis

Solution. The given equation is

$$\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} - 11\frac{dy}{dx} - 4y = 0$$

-1	1	-1	-9	-11	-4
	$-$	-1	2	7	4
	1	-2	-7	-4	$\boxed{0}$

$D = -1$

The equation in symbolic form is

$$(D^4 - D^3 - 9D^2 - 11D - 4)\,y = 0$$

Auxiliary equation is

$$D^4 - D^3 - 9D^2 - 11D - 4 = 0$$

$$(D + 1)^3 (D - 4) = 0$$

$D^3 - 2D^2 - 7D - 4 = 0$

-1	1	-2	-7	-4
		-1	3	4
	1	-3	-4	$\boxed{0}$

i.e.,

$\therefore\ D = -1, -1, -1, 4$

Here, we have three equal roots and one different root.

Hence the complete solution is $y = (c_1 + c_2x + c_3x^2)\,e^{-x} + c_4 e^{4x}$.

$D^2 - 4D + D - 4 = 0$

$D(D - 4) + 1(D - 4)$

$D^2 - 3D - 4 = 0$

$(D-4)(D+1)$

Example 5.

Solve the differential equation $(D^4 + 5D^2 + 6)\,y = 0.$ [K.U. 2016]

Solution. The given equation is

$$(D^4 + 5D^2 + 6)\,y = 0$$

$D^2 = t.$

The auxiliary equation is

$$D^4 + 5D^2 + 6 = 0$$

i.e.,

$$(D^2 + 3)(D^2 + 2) = 0$$

$t^2 + 5t + 6$

$t^2 + 3t + 2t + 6$

$t(t + 3) + 2(t + 3)$ so

$t = -2, -3.$

$D^2 = -2, -3.$

$\therefore\ D = \pm\sqrt{3}\,i, \pm\sqrt{2}\,i$

So, we have two pairs of complex roots, $D = \pm\sqrt{3}\,i, \pm\sqrt{2}\,i$

Hence, the complete solution is

$D = \alpha \pm i\beta$

$$y = c_1 \cos\sqrt{3}\,x + c_2 \sin\sqrt{3}\,x + c_3 \cos\sqrt{2}\,x + c_4 \sin\sqrt{2}\,x.$$

$D^2 = -2,\ \pm\sqrt{2}\,i$

$D^2 = -3 = \pm\sqrt{3}\,i$

Example 6.

Solve the differential equation $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = 0.$

Solution. The given equation is $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = 0$

Writing the equation in symbolic form, we have

$$(D^2 - 4D + 1)\,y = 0$$

The auxiliary equation is $D^2 - 4D + 1 = 0$

$$\therefore \quad D = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

Hence the complete solution is $y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x}$

D → real or complex
D = 2 ± √3 → irrational radical real
$y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x}$
II $y = e^{2x}[c_1 \cosh \sqrt{3}x + c_2 \sinh \sqrt{3}x]$

Another form :

$$y = e^{2x}\,[c_1 (\cosh \sqrt{3}\,x + \sinh \sqrt{3}\,x) + c_2 (\cosh \sqrt{3}\,x - \sinh \sqrt{3}\,x)]$$

or
$$y = e^{2x}\,[(c_1 + c_2) \cosh \sqrt{3}x + (c_1 - c_2) \sinh \sqrt{3}x]$$

or
$$y = e^{2x}\,[A \cosh \sqrt{3}x + B \sinh \sqrt{3}x].$$

$y = \alpha \pm i\beta$. Imaginary
$y = e^{\alpha x}[c_1 \cos \beta x + c_2 \sin \beta x]$

Example 7.

Solve the differential equation $\dfrac{d^4y}{dx^4} + a^4 y = 0.$ [K.U. 2018; M.D.U. 2017]

Solution. The given equation is $\dfrac{d^4y}{dx^4} + a^4 y = 0$

The symbolic form of the equation is

$$(D^4 + a^4)\,y = 0$$

Auxiliary equation is
$$D^4 + a^4 = 0$$

or
$$(D^2 + a^2)^2 - 2D^2a^2 = 0$$

or
$$(D^2 + a^2)^2 - (\sqrt{2}\,Da)^2 = 0$$

or
$$(D^2 - \sqrt{2}\,Da + a^2)(D^2 + \sqrt{2}\,Da + a^2) = 0$$

$$\therefore \quad D = \frac{-a}{\sqrt{2}} \pm i\frac{a}{\sqrt{2}},\ \frac{a}{\sqrt{2}} \pm i\frac{a}{\sqrt{2}}$$

(I) $D^2 + \sqrt{2}aD + a^2 = 0$
$D = \dfrac{-\sqrt{2}a \pm \sqrt{2a^2 - 4a^2}}{2}$
$D = \dfrac{-\sqrt{2}a \pm \sqrt{-2a^2}}{2}$
$D = \dfrac{\sqrt{2}a(-1 \pm i)}{2}$
$= \dfrac{-a}{\sqrt{2}} \pm \dfrac{a}{\sqrt{2}}i$

Hence, we have two pairs of complex roots, *viz.*, $\dfrac{-a}{\sqrt{2}} \pm i\dfrac{a}{\sqrt{2}},\ \dfrac{a}{\sqrt{2}} \pm i\dfrac{a}{\sqrt{2}}$

Hence the general solution is

$$y = e^{-\frac{a}{\sqrt{2}}x}\left(c_1 \cos \frac{a}{\sqrt{2}}x + c_2 \sin \frac{a}{\sqrt{2}}x\right) + e^{\frac{a}{\sqrt{2}}x}\left(c_3 \cos \frac{a}{\sqrt{2}}x + c_4 \sin \frac{a}{\sqrt{2}}x\right)$$

or
$$y = e^{\frac{-a}{\sqrt{2}}x} c_1 \cos\left(\frac{a}{\sqrt{2}}x + c_2\right) + e^{\frac{a}{\sqrt{2}}x} c_3 \cos\left(\frac{a}{\sqrt{2}}x + c_4\right).$$

EXERCISE 4.1

Solve the following differential equations [Q. 1 – 10] :

1. $\dfrac{d^2y}{dx^2} + (a+b)\dfrac{dy}{dx} + aby = 0$

2. $\dfrac{d^3y}{dx^3} - 13\dfrac{dy}{dx} - 12y = 0$

3. $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} - 6y = 0$

4. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0$

5. $\dfrac{d^4y}{dx^4} + 5\dfrac{d^3y}{dx^3} + 6\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} - 8y = 0$

6. $(D^3 - 2D^2 - 4D + 8)\,y = 0$

7. $\dfrac{d^2y}{dx^2} + \mu^2 y = 0$

8. $\dfrac{d^4y}{dx^4} + 4y = 0$

9. $\dfrac{d^4y}{dx^4} + 13\dfrac{d^2y}{dx^2} + 36y = 0.$

10. $\dfrac{d^6y}{dx^6} + 6\dfrac{d^4y}{dx^4} + 12\dfrac{d^2y}{dx^2} + 8y = 0.$

11. If $\dfrac{d^4y}{dx^4} - a^4y = 0$, show that $y = c_1 \cos ax + c_2 \sin ax + c_3 \cosh ax + c_4 \sinh ax.$

12. Solve $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 3x = 0$, given that for $t = 0, x = 0$ and $\dfrac{dx}{dt} = 12$.

13. Solve $\dfrac{d^2y}{dx^2} + y = 0$, given that $y = 2$ for $x = 0$ and $y = -2$ for $x = \dfrac{\pi}{2}$.

ANSWERS

1. $y = c_1 e^{-ax} + c_2 e^{-bx}$.
2. $y = c_1 e^{-x} + c_2 e^{-3x} + c_3 e^{4x}$
3. $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$
4. $y = (c_1 + c_2 x)\, e^{2x}$
5. $y = c_1 e^{x} + (c_2 + c_3 x + c_4 x^2)\, e^{-2x}$
6. $y = (c_1 + c_2 x)\, e^{2x} + c_3 e^{-2x}$
7. $y = c_1 \cos \mu x + c_2 \sin \mu x$
8. $y = e^{-x}(c_1 \cos x + c_2 \sin x) + e^{x}(c_3 \cos x + c_4 \sin x)$
9. $y = c_1 \cos 2x + c_2 \sin 2x + c_3 \cos 3x + c_4 \sin 3x$
10. $y = (c_1 + c_2 x + c_3 x^2) \cos \sqrt{2}x + (c_4 + c_5 x + c_6 x^2) \sin \sqrt{2}x$
12. $x = -6e^{-3t} + 6e^{-t}$
13. $y = 2 \cos x - 2 \sin x$.

$$= \frac{1}{\phi(a)} e^{ax} \int 1\,dx = \frac{1}{\phi(a)} \cdot e^{ax} \cdot x$$

$$\therefore \quad \frac{1}{f(D)} e^{ax} = x \cdot \frac{1}{\phi(a)} \cdot e^{ax} \qquad ...(2)$$

Differentiating (1) w.r.t. D, we get

$$f'(D) = \phi(D) + (D - a)\,\phi'(D)$$

$$\therefore \quad f'(a) = \phi(a)$$

From (2),

$$\frac{1}{f(D)} e^{ax} = \frac{xe^{ax}}{f'(a)}$$

$$= x \cdot \frac{1}{f'(D)} e^{ax} = x \cdot \frac{1}{\frac{d}{dD}[f(D)]} e^{ax}$$

$$\therefore \quad \frac{1}{f(D)} \cdot e^{ax} = x \cdot \frac{1}{\frac{d}{dD}[f(D)]} \cdot e^{ax} \text{ if } f(a) = 0.$$

Note.

If by using the above rule the denominator vanishes, repeat the rule again and so on.

SOLVED EXAMPLES

Example 1.

C.f → Complementary Soln
P.I → particular Integerat

Solve the differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 104e^{3x}. \quad \text{[K.U. 2013, 03; M.D.U. 2007]}$$

Solution. The given equation is $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 25y = 104e^{3x}$

The symbolic form of the equation is

$$(D^2 + 6D + 25)\,y = 104\,e^{3x}$$

Auxiliary equation is $D^2 + 6D + 25 = 0$

$$\therefore \quad D = \frac{-6 \pm \sqrt{36 - 100}}{2}$$

or $P.I = \frac{1}{f(D)} x$

$= \frac{1}{f(D)} e^{ax}$

and $D = a$

$$D = \frac{-6 \pm \sqrt{-64}}{2} = \frac{-6 \pm 8i}{2} = -3 \pm 4i$$

$$\therefore \quad \text{C.F.} = e^{-3x}\,[c_1 \cos 4x + c_2 \sin 4x]$$

$$\text{P.I.} = \frac{1}{D^2 + 6D + 25} \cdot 104\, e^{3x} = \frac{1}{9 + 18 + 25} \cdot 104\, e^{3x} \qquad \text{[Putting D = 3]}$$

$$= \frac{104 e^{3x}}{52} = 2e^{3x}$$

Thus, the complete solution is $y = \text{C.F.} + \text{P.I.}$

or $y = e^{-3x}\,(c_1 \cos 4x + c_2 \sin 4x) + 2e^{3x}.$

Example 2.

Solve the differential equation $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{3x}$.

Solution. The given equation is $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{3x}$ ...(1)

The symbolic form of the equation is

$$(D^2 - 6D + 9)\, y = e^{3x}$$

Auxiliary equation is $D^2 - 6D + 9 = 0$

or $(D-3)^2 = 0 \Rightarrow D = 3, 3$

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{3x}$

and P.I. $= \frac{1}{(D-3)^2} \cdot e^{3x} = \frac{1}{(3-3)^2} \cdot e^{3x}$ [*Putting D = 3*]

$= \frac{1}{0} \cdot e^{3x}$, which is a case of failure

$$\therefore \quad \text{P.I.} = \frac{1}{(D-3)^2} e^{3x} = x \cdot \frac{1}{\frac{d}{dD}(D-3)^2} \cdot e^{3x} = \frac{x}{2} \cdot \frac{1}{D-3} \cdot e^{3x}$$

derivative.

$$= \frac{x}{2}\left[\frac{1}{3-3} e^{3x}\right] \qquad \text{[Putting D = 3]}$$

$$= \frac{x}{2}\left[\frac{1}{0} e^{3x}\right] \qquad \text{[Again a case of failure]}$$

$$\text{P.I.} = \frac{x}{2} \cdot x \cdot \frac{1}{\frac{d}{dD}(D-3)} e^{3x} = \frac{x^2}{2} \cdot e^{3x}$$

Hence the complete solution is $\quad y = \text{C.F.} + \text{P.I.}$

i.e.,
$$y = (c_1 + c_2 x)\, e^{3x} + \frac{x^2}{2} e^{3x}.$$

Example 3. ***Solve the differential equation*** $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = e^{4x}$

Solution. The given equation is $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = e^{4x}$

The symbolic form of the equation is $(D^2 - 5D + 6)\, y = e^{4x}$

∴ Auxiliary equation is $\quad D^2 - 5D + 6 = 0$

or
$$(D-2)(D-3) = 0 \quad \Rightarrow \quad D = 2, 3$$

∴
$$\text{C.F.} = c_1 e^{2x} + c_2 e^{3x}$$

and
$$\text{P.I.} = \frac{1}{D^2 - 5D + 6} e^{4x} = \frac{1}{(D-2)(D-3)} e^{4x}$$

D = 4

Acc to Partial Frac

$$= \left[\frac{1}{D-3} - \frac{1}{D-2}\right] e^{4x} \qquad \textit{[Resolving into partial fractions]}$$

$\frac{1}{(D-3)(D-2)} = \frac{A}{(D-2)} - \frac{B}{(D-3)}$

$\frac{1}{D-\alpha} X = e^{\alpha x} \int e^{-\alpha x} x \, dx$

$$= \frac{1}{D-3} e^{4x} - \frac{1}{D-2} e^{4x}$$

$1 = A(D-3) - B(D-2)$

$$= e^{3x} \int e^{-3x} \cdot e^{4x}\, dx - e^{2x} \int e^{-2x} \cdot e^{4x}\, dx$$

D = 2

$1 = A(2-3) - B(2-2)$

$$= e^{3x} \int e^{x}\, dx - e^{2x} \int e^{2x}\, dx = e^{3x} \cdot e^{x} - e^{2x} \cdot \frac{e^{2x}}{2}$$

$-1 = A$

$B = 1$

∴
$$\text{P.I.} = e^{4x} - \frac{1}{2} e^{4x} = \frac{1}{2} e^{4x}$$

Hence, the complete solution is $\quad y = \text{C.F.} + \text{P.I.}$

i.e.,
$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{2} e^{4x}.$$

Imp

Example 4. ***Solve the differential equation*** $\dfrac{d^2y}{dx^2} + a^2 y = \sec ax.$ [K.U. 2012, 09, 05]

Solution. The given equation is $\dfrac{d^2y}{dx^2} + a^2 y = \sec ax$

The symbolic form of the equation is

$$(D^2 + a^2)\, y = \sec ax$$

$\therefore$ Auxiliary equation is $D^2 + a^2 = 0$ $\quad \frac{1}{D-\alpha} y = e^{\alpha x}\int e^{-\alpha x} x\,dx$

i.e., $D^2 = -a^2$

i.e., $D = \pm ia = 0 \pm ia$

$\therefore$ $\text{C.F.} = e^{0x}(c_1 \cos ax + c_2 \sin ax)$

$= c_1 \cos ax + c_2 \sin ax$ $\quad \frac{1}{D^2-(ia)^2}$

Now, $\text{P.I.} = \frac{1}{D^2+a^2} . \sec ax = \frac{1}{(D+ai)(D-ai)} \sec ax$

Acc to partial fraction [*Resolving into partial fractions*]

i.e., $\text{P.I.} = \frac{1}{2ai}\left[\frac{1}{D-ai} - \frac{1}{D+ai}\right] \sec ax$

$= \frac{1}{2ai}\left[\frac{1}{D-ai} \sec ax - \frac{1}{D+ai} \sec ax\right]$

By Euler theorem $e^{i\theta} = \cos\theta + i\sin\theta$

$= \frac{1}{2ai}\left[e^{iax}\int \frac{e^{-iax}}{\cos ax} dx - e^{-iax}\int \frac{e^{iax}}{\cos ax} dx\right]$ [*Using theorem, Art 4.1*]

$= \frac{1}{2ai} e^{iax}\int \frac{\cos ax - i \sin ax}{\cos ax} dx - \frac{1}{2ai} e^{-iax}\int \frac{\cos ax + i \sin ax}{\cos ax} dx$

[$\because$ $e^{\pm i\theta} = \cos\theta \pm i \sin\theta$ (*Euler's theorem*)]

$= \frac{1}{2ai} e^{iax}\int (1 - i \tan ax)\, dx - \frac{e^{-iax}}{2ai}\int (1 + i \tan ax)\, dx$

$= \frac{1}{2ai} . e^{iax}\left[x + \frac{i}{a}\log \cos ax\right] - \frac{e^{-iax}}{2ia}\left[x - \frac{i}{a}\log \cos ax\right]$

$= \frac{1}{2ai}[x(e^{iax} - e^{-iax})] + \frac{1}{2a^2}\log \cos ax\,(e^{iax} + e^{-iax})$

$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$= \frac{x}{a}\left[\frac{e^{iax} - e^{-iax}}{2i}\right] + \frac{1}{a^2}\log \cos ax\left[\frac{e^{iax} + e^{-iax}}{2}\right]$

$\sin\theta = \frac{e^{i\theta} - e^{i\theta}}{2}$

$= \frac{x}{a}\sin ax + \frac{1}{a^2}\log(\cos ax)\cos ax$

$\left[\because \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \text{ and } \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}\right]$

$= \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax . \log \cos ax$

Hence, the complete solution is

$$y = c_1 \cos ax + c_2 \sin ax + \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax \log(\cos ax).$$

Example 5. ***Solve the differential equation***

$$\frac{d^3y}{dx^3} + y = 3 + e^{-x} + 5e^{2x}$$ *[M.D.U. 2013; K.U. 2013]*

Solution. The given equation is $\frac{d^3y}{dx^3} + y = 3 + e^{-x} + 5e^{2x}$

The symbolic form of the equation is

$$(D^3 + 1)\,y = 3 + e^{-x} + 5e^{2x}$$

$\therefore$ Auxiliary equation is

$$D^3 + 1 = 0$$

$$(D + 1)(D^2 - D + 1) = 0$$

or y = 0, D = −1, D = 2.

$$D = -1;\ \frac{1 \pm \sqrt{1-4}}{2} = -1,\ \frac{1}{2} \pm i\,\frac{\sqrt{3}}{2}$$

$\therefore \frac{3}{D+1}, +\frac{1}{D+1}e^{-x}$

$\therefore \frac{5e^{2x}}{D^3+1}$

$$\text{C.F.} = c_1e^{-x} + e^{x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right)$$

$$\text{P.I.} = \frac{1}{D^3+1}(3 + e^{-x} + 5e^{2x})$$

$= 3 + \frac{1}{3}e^{-x} + \frac{5}{9}e^{2x}$

$$= 3\cdot\frac{1}{D^3+1}\cdot e^{0x} + \frac{1}{D^3+1}\cdot e^{-x} + 5\cdot\frac{1}{D^3+1}\cdot e^{2x}$$

$$= 3\cdot\frac{1}{0+1}\cdot e^{0x} + \frac{1}{(-1)^3+1}\cdot e^{-x} + 5\cdot\frac{1}{2^3+1}e^{2x}$$

$$= 3 + \frac{1}{0}e^{-x} + \frac{5}{9}e^{2x}$$ *[Case of failure]*

$$= 3 + x\cdot\frac{1}{\frac{d}{dD}(D^3+1)}e^{-x} + \frac{5}{9}e^{2x} = 3 + x\cdot\frac{1}{3D^2}\cdot e^{-x} + \frac{5}{9}e^{2x}$$

$$= 3 + x\cdot\frac{1}{3(-1)^2}\cdot e^{-x} + \frac{5}{9}e^{2x} = 3 + \frac{xe^{-x}}{3} + \frac{5}{9}e^{2x}$$

Hence the complete solution is

$$y = c_1e^{-x} + e^{x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right) + 3 + \frac{xe^{-x}}{3} + \frac{5}{9}e^{2x}.$$

Example 6. ***Solve the differential equation*** $\frac{d^2y}{dx^2} - y = \cosh x$. *[M.D.U. 2017]*

Solution. The given equation is $\frac{d^2y}{dx^2} - y = \cosh x$

or $(D^2 - 1)\,y = \cosh x$

$\therefore$ Auxiliary equation is $D^2 - 1 = 0 \Rightarrow D^2 = 1 \Rightarrow D = \pm 1$

$\therefore$ $$\text{C.F} = c_1 e^x + c_2 e^{-x}$$

and $$\text{P.I.} = \frac{1}{D^2-1}\cosh x = \frac{1}{D^2-1}\left(\frac{e^x+e^{-x}}{2}\right)$$

$\cosh x = \left(\frac{e^x + e^{-x}}{2}\right)$

$$= \frac{1}{2}\cdot\frac{1}{D^2-1}e^x + \frac{1}{2}\cdot\frac{1}{D^2-1}e^{-x}$$

$$= \frac{1}{2}\cdot\frac{1}{0}e^x + \frac{1}{2}\cdot\frac{1}{0}e^{-x}, \text{ which is a case of failure.}$$

$\therefore$ $$\text{P.I.} = \frac{x}{2}\cdot\frac{1}{\frac{d}{dD}(D^2-1)}e^x + \frac{x}{2}\frac{1}{\frac{d}{dD}(D^2-1)}e^{-x}$$

$$= \frac{x}{2}\cdot\frac{1}{2D}e^x + \frac{x}{2}\cdot\frac{1}{2D}e^{-x}$$

$$= \frac{x}{4}e^x - \frac{x}{4}e^{-x}$$

Hence, the complete solution is y = C.F. + P.I.

i.e., $$y = c_1e^x + c_2e^{-x} + \frac{x}{4}(e^x - e^{-x}).$$

Example 7. *If* $\frac{d^2x}{dt^2} + \frac{g}{b}(x-a) = 0$, $(a > 0, b > 0, g > 0)$ *and* $x = \alpha$, $\frac{dx}{dt} = 0$, *when* t=

show that, $x = a + (\alpha - a)\cos\sqrt{\frac{g}{b}}\,.t$.

Solution. The given equation is $\frac{d^2x}{dt^2} + \frac{g}{b}(x-a) = 0$

or Independent Varible. $\frac{d^2x}{dt^2} + \frac{g}{b}x - \frac{ga}{b} = 0$, $\frac{d^2x}{dt^2} + \frac{g}{b}x = \frac{ga}{b}$.

The symbolic form of the equation is

$$\left(D^2 + \frac{g}{b}\right)x = \frac{ga}{b}, \text{ where } D = \frac{d}{dt}$$

Auxiliary equation is $D^2 + \frac{g}{b} = 0$

$\therefore$ $$D = \pm i\sqrt{\frac{g}{b}} = 0 \pm i\sqrt{\frac{g}{b}}$$

$\therefore$ $$\text{C.F.} = e^{0.t}\left(c_1\cos\sqrt{\frac{g}{b}}\,.t + c_2\sin\sqrt{\frac{g}{b}}\,.t\right)$$

$$= c_1 \cos \sqrt{\frac{g}{b}} \cdot t + c_2 \sin \sqrt{\frac{g}{b}} \cdot t$$

$$\text{P.I.} = \frac{1}{D^2 + \frac{g}{b}} \cdot \frac{ga}{b} = \frac{ga}{b} \cdot \frac{1}{D^2 + \frac{g}{b}} \cdot e^{0.t}$$

$$= \frac{ga}{b} \cdot \frac{1}{0 + \frac{g}{b}} \cdot e^{0.t} = a$$

Hence the complete solution is

$$x = c_1 \cos \sqrt{\frac{g}{b}} \cdot t + c_2 \sin \sqrt{\frac{g}{b}} \cdot t + a \quad \text{...(1)}$$

When $t = 0$, $x = \alpha$ [*Given*]

$\therefore$ From (1), $\alpha = c_1 \cos 0 + c_2 \sin 0 + a$

$\alpha = c_1 + a \Rightarrow c_1 = \alpha - a$

Differentiating (1) w.r.t. 't', we have

$$\frac{dx}{dt} = -c_1 \sqrt{\frac{g}{b}} \sin \sqrt{\frac{g}{b}} \cdot t + c_2 \sqrt{\frac{g}{b}} \cos \sqrt{\frac{g}{b}} \cdot t \quad \text{...(2)}$$

When $t = 0$, $\frac{dx}{dt} = 0$ [*Given*]

$\therefore$ From (2), $0 = c_2 \sqrt{\frac{g}{b}} \Rightarrow c_2 = 0$

Substituting the values of c_1 and c_2 in (1), we have

$$x = a + (\alpha - a) \cos\left(\sqrt{\frac{g}{b}} \cdot t\right),$$ which is the required result.

EXERCISE 4.2

Solve the following differential equations [Q. 1 – 12] :

1. $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 3y = e^{2x}$
2. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{2x}$
3. $\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + k^2y = e^{x}$
4. $2\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + y = e^{x} + 1$ [K.U. 2000]
5. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{2x} + e^{-2x}$

6. (i) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=2\sinh 2x$ (ii) $(D^2-3D+2)y=\cosh x$

7. $\frac{d^2y}{dx^2}-7\frac{dy}{dx}+6y=e^{2x}$, given that $y=0$ when $x=0$. [M.D.U. 2014]

8. $\frac{d^3y}{dx^3}-5\frac{d^2y}{dx^2}+7\frac{dy}{dx}-3y=e^{2x}\cosh x$ [K.U. 2016]

9. $\frac{d^3y}{dx^3}-6\frac{d^2y}{dx^2}+11\frac{dy}{dx}-6y=e^{2x}$

10. $\frac{d^3y}{dx^3}-2\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=e^{3x}$

11. $\frac{d^2y}{dx^2}+y=\operatorname{cosec} x$ [K.U. 2017, 15; M.D.U. 2015]

12. $\frac{d^2y}{dx^2}+y=\sec x$ [M.D.U. 2011; K.U. 2004]

13. $\frac{d^2y}{dx^2}+16y=\sec 4x$ [M.D.U. 2014]

14. $(D^2-3D+2)y=\cos e^{-x}$ [M.D.U. 2007]

15. Prove that $\frac{1}{D+a}X=e^{-ax}\int(e^{ax}.\ X)\,dx$, where X is function of x. [M.D.U. 2016]

[**Hint :** Proceed as in Art. 4.10, page 4.14].

ANSWERS

1. $y=c_1e^{\frac{x}{2}}+c_2e^{-\frac{3}{2}x}+\frac{1}{21}e^{2x}$

2. $y=c_1e^{x}+c_2e^{2x}+x\,e^{2x}$

3. $y=(c_1+c_2x)e^{kx}+\frac{1}{(1-k)^2}.e^{x}$

4. $y=(c_1+c_2x)e^{x}+c_3e^{-\frac{x}{2}}+\frac{1}{6}x^2e^{x}+1$

5. $y=(c_1+c_2x)e^{-2x}+\frac{e^{2x}}{16}+\frac{1}{2}x^2e^{-2x}$

6. (i) $y=(c_1+c_2x)e^{-2x}+\frac{e^{2x}}{16}-\frac{x^2}{2}e^{-2x}$ (ii) $y=c_1e^{x}+c_2e^{2x}-\frac{x}{2}e^{x}+\frac{1}{12}e^{-x}$

7. $y=c_1(e^{x}-e^{6x})+\frac{1}{4}(e^{6x}-e^{2x})$

8. $y=e^{x}(c_1+c_2x)+c_3e^{3x}+\frac{x}{8}e^{3x}-\frac{1}{8}x^2e^{x}$

9. $y=c_1e^{x}+c_2e^{2x}+c_3e^{3x}-x\,e^{2x}$

10. $y=c_1e^{-2x}+c_2e^{x}+c_3e^{3x}+\frac{x}{10}e^{3x}$

11. $y=c_1\cos x+c_2\sin x+\sin x\log\sin x-x\cos x$

12. $y=c_1\cos x+c_2\sin x+x\sin x+\cos x\log\cos x$

13. $y=c_1\cos 4x+c_2\sin 4x+\frac{x}{4}\sin 4x+\frac{1}{16}\cos 4x\log(\cos 4x)$.

14. $y=c_1e^{x}+c_2e^{2x}-e^{2x}\cos e^{-x}$.

$$= \frac{1}{-a^2 - \beta^2} (a \cos ax - \beta \sin ax)$$

Thus, the particular integral in case of sin ax and cos ax can be completely evaluated.

SOLVED EXAMPLES

Example 1.

Solve the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x.$ [K.U. 2001]

Solution. The given equation is $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x$

$D^2 = -a^2$

$D^2 = -(2)^2 = -4$

The symbolic form of the equation is

$$(D^2 + D + 1)\, y = \sin 2x.$$

$\therefore$ Auxiliary equation is $\quad D^2 + D + 1 = 0$

$$\therefore \quad D = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1}{2} \pm i\,\frac{\sqrt{3}}{2}$$

$$\therefore \quad \text{C.F.} = e^{-x/2}\left[c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right]$$

$$\text{P.I.} = \frac{1}{D^2 + D + 1} \sin 2x$$

$$= \frac{1}{-2^2 + D + 1} \sin 2x \qquad [\textit{Put } D^2 = -2^2]$$

$$= \frac{1}{D-3} \sin 2x$$

$$= \frac{D+3}{(D-3)(D+3)} \sin 2x = \frac{D+3}{D^2 - 9} \sin 2x$$

$$= \frac{D+3}{-2^2 - 9} \sin 2x \qquad [\because D^2 = -2^2]$$

$D = \frac{d}{dx}$

$$= -\frac{1}{13}(D+3)\sin 2x = -\frac{1}{13}[D \sin 2x + 3 \sin 2x]$$

$$= -\frac{1}{13}\left[\frac{d}{dx}\sin 2x + 3 \sin 2x\right]$$

$$= -\frac{1}{13}[2\cos 2x + 3\sin 2x]$$

Hence the complete solution is

$$y = e^{-x/2}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) - \frac{1}{13}(2\cos 2x + 3\sin 2x).$$

Example 2.

Solve the differential equation

$$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x.$$ [K.U. 201

Solution. The given equation is $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x.$

The symbolic form of the equation is $(D^3 + D^2 + D + 1)\,y = \sin 2x$

Auxiliary equation is $D^3 + D^2 + D + 1 = 0$

or $D^2(D + 1) + (D + 1) = 0$

or $(D^2 + 1)(D + 1) = 0 \Rightarrow D = \pm i, -1$

$\therefore$ C.F. $= c_1 e^{-x} + c_2 \cos x + c_3 \sin x$

Now, P.I. $= \dfrac{1}{D^3 + D^2 + D + 1}\sin 2x = \dfrac{1}{(D+1)(D^2+1)}.\sin 2x$

$= \dfrac{1}{(D+1)(-2^2+1)}.\sin 2x$ [Put $D^2 = -2^2$]

$= -\dfrac{1}{3}.\dfrac{1}{D+1}.\sin 2x = -\dfrac{1}{3}.\dfrac{D-1}{(D+1)(D-1)}.\sin 2x$

$= -\dfrac{1}{3}.\dfrac{D-1}{D^2-1}.\sin 2x = -\dfrac{1}{3}.\dfrac{D-1}{-4-1}.\sin 2x$ [Put $D^2 = -2^2$]

$= \dfrac{1}{15}(D-1)\sin 2x = \dfrac{1}{15}[D\sin 2x - \sin 2x]$

$= \dfrac{1}{15}[2\cos 2x - \sin 2x]$

$\dfrac{1}{(-4)D + (-4) + D + 1}\sin 2x$

$\dfrac{1}{-3D-3}$

$= \dfrac{-1}{3}\dfrac{1}{(D+1)}$

Hence the complete solution is

$$y = c_1 e^{-x} + c_2 \cos x + c_3 \sin x + \frac{1}{15}[2\cos 2x - \sin 2x].$$

Example 3.

Solve the differential equation

$$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = \cos 2x.$$

Solution. The given equation is $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = \cos 2x$

The symbolic form of the equation is

$$(D^3 + D^2 - D - 1)\,y = \cos 2x$$

Auxiliary equation is $\quad D^3 + D^2 - D - 1 = 0$

or $\quad D^2(D + 1) - (D + 1) = 0$

or $\quad (D + 1)(D^2 - 1) = 0$

or $\quad (D + 1)^2 (D - 1) = 0$

i.e., $\quad D = -1, -1, 1$

$\therefore \quad$ C.F. $= c_1 e^x + (c_2 + c_3 x)\, e^{-x}$

Now, $\quad$ P.I. $= \frac{1}{D^3 + D^2 - D - 1} . \cos 2x = \frac{1}{(D+1)(D^2-1)} . \cos 2x$

$$= \frac{1}{(D+1)(-4-1)} . \cos 2x \qquad [\textit{Put } D^2 = -2^2 = -4]$$

$$= -\frac{1}{5} . \frac{1}{D+1} . \cos 2x = -\frac{1}{5} . \frac{D-1}{(D+1)(D-1)} . \cos 2x$$

$$= -\frac{1}{5} . \frac{D-1}{D^2-1} . \cos 2x$$

$$= -\frac{1}{5} . \frac{D-1}{-4-1} . \cos 2x = \frac{1}{25}(D-1)\cos 2x$$

$$= \frac{1}{25}(D\cos 2x - \cos 2x)$$

$$= \frac{1}{25}(-2\sin 2x - \cos 2x) \qquad \left[\because D = \frac{d}{dx}\right]$$

$$= -\frac{1}{25}(2\sin 2x + \cos 2x)$$

Hence the complete solution is

$$y = c_1 e^x + (c_2 + c_3 x)\, e^{-x} - \frac{1}{25}(2\sin 2x + \cos 2x).$$

Example 4.

Solve the differential equation $\frac{d^2y}{dx^2} - 4y = e^x + \sin 2x$.

Solution. The given equation is $\frac{d^2y}{dx^2} - 4y = e^x + \sin 2x$

The symbolic form of the equation is $(D^2 - 4)\, y = e^x + \sin 2x$

Auxiliary equation is $D^2 - 4 = 0$

$$D = \pm 2$$

$\therefore$ $$\text{C.F.} = c_1 e^{2x} + c_2 e^{-2x}$$

Now, $$\text{P.I.} = \frac{1}{D^2-4} \cdot (e^x + \sin 2x) = \frac{1}{D^2-4} \cdot e^x + \frac{1}{D^2-4} \sin 2x$$

$$= \frac{1}{1^2-4} \cdot e^x + \frac{1}{-4-4} \cdot \sin 2x \quad \left[\because \frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax\right]$$

$$= -\frac{1}{3} e^x - \frac{1}{8} \sin 2x$$

Hence, the complete or general solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{3} e^x - \frac{1}{8} \sin 2x.$$

Example 5.

Solve the differential equation $\frac{d^2y}{dx^2} + y = \sin x \sin 2x.$ [K.U. 20

$(D^2+1)y = \frac{1}{2}(2\sin x \cdot \sin$

$2\sin A \cdot \sin B =$

$(\cos(A-B) - \cos($

Solution. The given equation is $\frac{d^2y}{dx^2} + y = \sin x \sin 2x.$

The symbolic form of the equation is $(D^2 + 1)\, y = \sin x \sin 2x$

Auxiliary equation is $D^2 + 1 = 0$

i.e., $$D = \pm i = 0 \pm i$$

$\therefore$ $$\text{C.F.} = e^{0x} (c_1 \cos x + c_2 \sin x) = c_1 \cos x + c_2 \sin x$$

Now, $$\text{P.I.} = \frac{1}{D^2+1} \cdot \sin x \sin 2x = \frac{1}{D^2+1}\left[\frac{1}{2}(2 \sin 2x \sin x)\right]$$

$$= \frac{1}{2} \cdot \frac{1}{D^2+1} (\cos x - \cos 3x) \quad [\because 2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

$$= \frac{1}{2}\left[\frac{1}{D^2+1}\cdot\cos x - \frac{1}{D^2+1}\cdot\cos 3x\right]$$

$$= \frac{1}{2}\left[x\cdot\frac{1}{\frac{d}{dD}(D^2+1)}\cdot\cos x - \frac{1}{-9+1}\cos 3x\right]$$

$$\left[\because \frac{1}{f(D^2)}\cos ax = x\cdot\frac{1}{\frac{d}{dD}[f(D^2)]}\cdot\cos ax \text{ if } f(-a^2)=0 \text{ (case of failure)}\right]$$

$$= \frac{1}{2}\left[x\cdot\frac{1}{2D}\cdot\cos x + \frac{1}{8}\cos 3x\right] = \frac{1}{2}\left[\frac{x}{2}\cdot\frac{1}{D}\cos x + \frac{1}{8}\cos 3x\right]$$

$$= \frac{1}{2}\left[\frac{x}{2}\int\cos x\,dx + \frac{1}{8}\cos 3x\right] = \frac{x}{4}\sin x + \frac{1}{16}\cos 3x$$

Hence, the complete solution is

$$y = c_1\cos x + c_2\sin x + \frac{x}{4}\sin x + \frac{1}{16}\cos 3x.$$

Example 6.

Solve the differential equation $\dfrac{d^2y}{dx^2} + a^2y = \sin ax$ *[M.D.U. 2009]*

Solution. The given equation is $\dfrac{d^2y}{dx^2} + a^2y = \sin ax$

The symbolic form of the equation is $(D^2+a^2)\,y = \sin ax$

Auxiliary equation is $D^2 + a^2 = 0$

i.e., $D = \pm ia = 0 \pm ia$

$\therefore$ $\text{C.F.} = e^{0x}(c_1\cos ax + c_2\sin ax) = c_1\cos ax + c_2\sin ax$

Now, $\text{P.I.} = \dfrac{1}{D^2+a^2}\sin ax$

$$= x\cdot\frac{1}{\frac{d}{dD}(D^2+a^2)}\cdot\sin ax \qquad \textit{[Case of failure]}$$

$$= x\cdot\frac{1}{2D}\cdot\sin ax = \frac{x}{2}\cdot\frac{1}{D}\sin ax$$

$$= \frac{x}{2}\int\sin ax\,dx$$

$$= \frac{x}{2}\left(-\frac{\cos ax}{a}\right) = \frac{-x\cos ax}{2a}$$

Hence the complete solution is $y = c_1 \cos ax + c_2 \sin ax - \frac{x}{2a} \cdot \cos ax.$

EXERCISE 4.3

***Solve the following differential equations* [Q. 1 – 5] :**

1. (i) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin 3x$ (ii) $(D^3 + 6D^2 + 11D + 6)y = 2\sin x$

(iii) $(D^3 - 2D^2 + 3)y = \cos x$ (iv) $(D^4 - 1)y = \sin 2x$ [K.U. 2014; M.D.U. 200..]

2. (i) $\frac{d^4y}{dx^4} + y = \cos x$ (ii) $\frac{d^4y}{dx^4} + 2n^2\frac{d^2y}{dx^2} + n^4y = \cos mx.$

3. (i) $\frac{d^4y}{dx^4} - a^4y = \sin ax$ (ii) $\frac{d^4y}{dx^4} - m^4y = \sin mx$ [K.U. 201..]

(iii) $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = \cos x$

4. (i) $(D^4 + 1)y = \sin^2 x$ (ii) $(D^2 + 1)(D^2 + 4)y = \cos\left(\frac{x}{2}\right)\cos\left(\frac{3x}{2}\right)$

(iii) $(D^2 + 1)(D^2 + 4)y = \cos\frac{x}{2}\sin\frac{x}{2}$ [M.D.U. 2017] (iv) $(D^2 - 4D + 3)y = \sin 3x \cos 2x$ [K.U. 2000]

(v) $(D^2 - 4D + 4)y = e^{-4x} + 5\cos 3x$

5. (i) $\frac{d^3y}{dx^3} + y = \sin 3x - \cos^2\frac{x}{2}$ (ii) $\frac{d^2y}{dx^2} + 4y = e^x + \sin 2x$

(iii) $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x + \cos x.$

6. Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y + 37\sin 3x = 0$ and find the value of y when $x = \frac{\pi}{2}$ if it is given that $y = ...$ and $\frac{dy}{dx} = 0$ when $x = 0$.

7. Solve $\frac{d^2x}{dt^2} + b^2x = k\cos bt,$ given that $x = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$.

ANSWERS

(i) $y = c_1 e^x + c_2 e^{2x} - \frac{7}{130} \sin 3x + \frac{9}{130} \cos 3x$

(ii) $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x} - \frac{\cos x}{5}$

(iii) $y = c_1 e^{-x} + e^{\frac{3}{2}x}\left[c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x\right] + \frac{1}{26}(5 \cos x - \sin x).$

(iv) $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x + \frac{\sin 2x}{15}$

(i) $y = e^{x/\sqrt{2}}\left[c_1 \cos \frac{x}{\sqrt{2}} + c_2 \sin \frac{x}{\sqrt{2}}\right] + e^{-x/\sqrt{2}}\left[c_3 \cos \frac{x}{\sqrt{2}} + c_4 \sin \frac{x}{\sqrt{2}}\right] + \frac{1}{2} \cos x$

(ii) $y = (c_1 + c_2 x) \cos nx + (c_3 + c_4 x) \sin nx + \frac{\cos mx}{(m^2 - n^2)^2}.$

(i) $y = c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax + \frac{x}{4a^3} \cos ax$

(ii) $y = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx + \frac{x}{4m^3} \cos mx$

(iii) $y = (c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x - \frac{x^2}{8} \cos x$

(i) $y = e^{\frac{1}{\sqrt{2}}x}\left[c_1 \cos \frac{x}{\sqrt{2}} + c_2 \sin \frac{x}{\sqrt{2}}\right] + e^{-\frac{1}{\sqrt{2}}x}\left[c_3 \cos \frac{x}{\sqrt{2}} + c_4 \sin \frac{x}{\sqrt{2}}\right] + \frac{1}{2} - \frac{1}{34} \cos 2x$

(ii) $y = (c_1 \cos x + c_2 \sin x) + (c_3 \cos 2x + c_4 \sin 2x) + \frac{x}{12}\left[\sin x - \frac{1}{2} \sin 2x\right]$

(iii) $y = (c_1 \cos x + c_2 \sin x) + (c_3 \cos 2x + c_4 \sin 2x) - \frac{x}{12} \cos x$

(iv) $y = c_1 e^x + c_2 e^{3x} + \frac{1}{884}(-11 \sin 5x + 10 \cos 5x) + \frac{1}{20}(\sin x + 2 \cos x)$

(v) $y = (c_1 + c_2 x)e^{2x} + \frac{1}{36} e^{-4x} - \frac{5}{169}(12 \sin 3x + 5 \cos 3x).$

(i) $y = c_1 e^{-x} + e^{x/2}\left[c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x\right] + \frac{1}{730}[\sin 3x + 27 \cos 3x] - \frac{1}{2} - \frac{1}{4}(\cos x - \sin x)$

(ii) $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} x \cos 2x + \frac{1}{5} e^x$.

(iii) $y = c_1 e^{x} + e^{x} (c_2 \cos x + c_3 \sin x) + x e^x + \frac{3}{10} \sin x + \frac{1}{10} \cos x$.

6. $y = e^{-x} (c_1 \cos 3x + c_2 \sin 3x) + 6 \cos 3x - \sin 3x \,;\, y = 1$.

7. $x = \frac{kt}{2b} \sin bt$.

4.12.3. Case III. To evaluate $\frac{1}{f(D)} x^m$, where m is a positive integer.

To evaluate the particular integral $\frac{1}{f(D)} . x^m$, where m is a +ve integer; from $f(D)$ take out the common lowest degree term and the remaining factor will be of the form $[1 \pm \phi(D)]$.

Take $[1 \pm \phi(D)]$ to the numerator with negative index and expand it by Binomial theorem. The expansion will be made only upto the terms where the index of D is m, because D^{m+1} when multiplied by x^m (*i.e.*, $D^{m+1} . x^m$) will be zero. Operate on x^m with each term.

Important Expansions. The following binomial expansions will be commonly used in these type of questions :

binomial

$(x+a)^n = {}^nC_0 x^n a^0 + {}^nC_1 x^{n-1} a^1 + {}^nC_2 x^{n-2} a^2 + \ldots$

1. $(1 - D)^{-1} = 1 + D + D^2 + D^3 + \ldots\ldots$ to ∞.
2. $(1 + D)^{-1} = 1 - D + D^2 - D^3 + \ldots\ldots$ to ∞.
3. $(1 - D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \ldots\ldots + (r + 1) D^r + \ldots.$ to ∞.
4. $(1 - D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + \ldots\ldots + \frac{(r+1)(r+2)}{2} D^r + \ldots\ldots$ to ∞.

SOLVED EXAMPLES

Example 1.

Solve the differential equation $\frac{d^2y}{dx^2} - 4y = x^2$.

Solution. The given equation is $\frac{d^2y}{dx^2} - 4y = x^2$

The symbolic form of the equation is $(D^2 - 4) y = x^2$

$\therefore$ Auxiliary equation is $D^2 - 4 = 0 \;\Rightarrow\; D = \pm 2$

$\therefore$ $\quad$ C.F. $= c_1 e^{2x} + c_2 e^{-2x}$

$$\text{P.I.} = \frac{1}{D^2 - 4} \cdot x^2$$

Now,

$$= \frac{1}{-4\left(1 - \frac{D^2}{4}\right)} x^2 = -\frac{1}{4}\left(1 - \frac{D^2}{4}\right)^{-1} \cdot x^2$$

$$= -\frac{1}{4}\left[1 + \frac{D^2}{4} + \ldots\right] x^2 \qquad \textit{[By Binomial theorem]}$$

$$= -\frac{1}{4}\left[x^2 + \frac{1}{4} D^2 (x^2)\right] \qquad \textit{[Retaining terms upto } D^2\textit{]}$$

$$= -\frac{1}{4}\left[x^2 + \frac{1}{4} \cdot 2\right] = -\frac{1}{4}\left(x^2 + \frac{1}{2}\right)$$

Hence, the complete solution is $y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4}\left(x^2 + \frac{1}{2}\right)$.

Example 2. ***Solve the differential equation*** $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \sin x$.

[K.U. 2019, 12]

Solution. The given equation is $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \sin x$

The symbolic form of the equation is

$$(D^2 + D - 2)\, y = x + \sin x$$

$\therefore$ Auxiliary equation is $\quad D^2 + D - 2 = 0$

$(D + 2)(D - 1) = 0 \quad \Rightarrow \quad D = 1, -2$

$$\text{C.F.} = c_1 e^{x} + c_2 e^{-2x}$$

Now,

$$\text{P.I.} = \frac{1}{D^2 + D - 2}(x + \sin x)$$

$$= \frac{1}{D^2 + D - 2} x + \frac{1}{D^2 + D - 2} \sin x$$

$$= \frac{1}{-2\left(1 - \frac{D}{2} - \frac{D^2}{2}\right)} \cdot x + \frac{1}{-1^2 + D - 2} \cdot \sin x$$

$$= -\frac{1}{2}\left(1 - \frac{D}{2} - \frac{D^2}{2}\right)^{-1} \cdot x + \frac{1}{D - 3} \cdot \sin x$$

$$= -\frac{1}{2}\left[1 - \left(\frac{D}{2} + \frac{D^2}{2}\right)\right]^{-1} \cdot x + \frac{D + 3}{(D - 3)(D + 3)} \cdot \sin x$$

$$= -\frac{1}{2}\left[1 + \left(\frac{D}{2} + \frac{D^2}{2}\right) + \ldots\ldots\right] x + \frac{D + 3}{D^2 - 9} \cdot \sin x$$

$$= -\frac{1}{2}\left[x + \frac{1}{2}D(x)\right] + \frac{D+3}{-1-9} \cdot \sin x$$

$D^2 x = 1$ two time derivate x,

$$= -\frac{1}{2}\left(x + \frac{1}{2}\right) - \frac{1}{10}(D+3) \cdot \sin x$$

$$= -\frac{1}{2}\left(x + \frac{1}{2}\right) - \frac{1}{10}(D \sin x + 3 \sin x)$$

$$= -\frac{1}{2}\left(x + \frac{1}{2}\right) - \frac{1}{10}(\cos x + 3 \sin x)$$

Hence, the complete solution is $y = c_1 e^x + c_2 e^{-2x} - \frac{1}{2}\left(x + \frac{1}{2}\right) - \frac{1}{10}(\cos x + 3 \sin x)$.

Example 3.

Solve the differential equation $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2 + e^x + \cos 2x$. [K.U. 2014]

Solution. The given equation is $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2 + e^x + \cos 2x$

The symbolic form of the equation is

$$(D^2 - 4D + 4)\,y = x^2 + e^x + \cos 2x$$

Auxiliary equation is

$$D^2 - 4D + 4 = 0$$

i.e., $(D-2)^2 = 0 \quad \Rightarrow \quad D = 2, 2$

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{2x}$

Now, P.I. $= \frac{1}{D^2 - 4D + 4}(x^2 + e^x + \cos 2x)$

$$= \frac{1}{D^2 - 4D + 4}x^2 + \frac{1}{D^2 - 4D + 4} \cdot e^x + \frac{1}{D^2 - 4D + 4}\cos 2x$$

$\frac{1}{4}\left[1 + \left(\frac{D^2 - 4D}{4}\right)\right]$

$$= \frac{1}{4\left(1 - D + \frac{D^2}{4}\right)}x^2 + \frac{1}{1^2 - 4.1 + 4}e^x + \frac{1}{-2^2 - 4D + 4}\cos 2x$$

$$= \frac{1}{4}\left[1 - \left(D - \frac{D^2}{4}\right)\right]^{-1} \cdot x^2 + e^x - \frac{1}{4}\left(\frac{1}{D}\cos 2x\right)$$

$$= \frac{1}{4}\left[1 + D - \frac{D^2}{4} + \left(D - \frac{D^2}{2}\right)^2 + \ldots\right]x^2 + e^x - \frac{1}{4}\int \cos 2x\, dx$$

$$= \frac{1}{4}\left[1 + D - \frac{D^2}{4} + D^2 + \ldots\right]x^2 + e^x - \frac{1}{4} \cdot \frac{\sin 2x}{2}$$

$$= \frac{1}{4}\left[1 + D + \frac{3}{4}D^2 + \ldots\right]x^2 + e^x - \frac{1}{8}\sin 2x$$

$$= \frac{1}{4}\left[x^2 + D(x^2) + \frac{3}{4}D^2(x^2)\right] + e^x - \frac{1}{8}\sin 2x$$

$$= \frac{1}{4}\left[x^2 + 2x + \frac{3}{2}\right] + e^x - \frac{1}{8}\sin 2x$$

Hence the complete solution is

$$y = (c_1 + c_2 x)e^{2x} + \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right) + e^x - \frac{1}{8}\sin 2x.$$

Example 4. ***Solve the differential equation***

$$(D^2 + 1)(D^2 + 4)\,y = \cos\frac{x}{2}\cos\frac{3x}{2} + x. \qquad \text{[M.D.U. 2016]}$$

Solution. The given equation is $(D^2 + 1)(D^2 + 4)\,y = \cos\left(\frac{x}{2}\right)\cos\left(\frac{3x}{2}\right) + x$

$\therefore$ Auxiliary equation is $(D^2 + 1)(D^2 + 4) = 0 \quad \Rightarrow \quad D = \pm i, \pm 2i$

$\therefore$ C.F. $= (c_1 \cos x + c_2 \sin x) + (c_3 \cos 2x + c_4 \sin 2x)$

Now,

$$\text{P.I.} = \frac{1}{(D^2+1)(D^2+4)}\left(\cos\frac{x}{2}\cos\frac{3x}{2} + x\right)$$

$$= \frac{1}{(D^2+1)(D^2+4)}\left[\frac{1}{2}\cdot 2\cos\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right) + x\right]$$

$$= \frac{1}{(D^2+1)(D^2+4)}\left[\frac{1}{2}(\cos 2x + \cos x) + x\right]$$

D → 1

$$= \frac{1}{2}\left[\frac{1}{(D^2+1)(D^2+4)}\cos 2x + \frac{1}{(D^2+1)(D^2+4)}\cos x\right] + \frac{1}{(D^2+1)(D^2+4)}x$$

$$= \frac{1}{0}\cos 2x + \frac{1}{0}\cos x + \frac{1}{D^4 + 5D^2 + 4}\cdot x \qquad \textit{[Case of failure]}$$

$$= \frac{x}{2}\left[\frac{1}{\frac{d}{dD}(D^4+5D^2+4)}\cos 2x + \frac{1}{\frac{d}{dD}(D^4+5D^2+4)}\cos x\right] + \frac{1}{4\left(1+\frac{5}{4}D^2+\frac{D^4}{4}\right)}\cdot x$$

→ D⁴ की पावर 2 से ज्यादा। D वाली छोड़ दी जाती

$$= \frac{x}{2}\left[\underset{\text{[Put } D^2 = -4]}{\frac{1}{4D^3+10D}}\cos 2x + \underset{\text{[Put } D^2 = -1]}{\frac{1}{4D^3+10D}}\cos x\right] + \frac{1}{4\left[1+\left(\frac{5}{4}D^2+\frac{D^4}{4}\right)\right]}\cdot x$$

$$= \frac{x}{2}\left[\frac{1}{-6D}\cos 2x + \frac{1}{6D}\cos x\right] + \frac{1}{4}\left[1+\left(\frac{5}{4}D^2+\frac{D^4}{4}\right)\right]^{-1}x$$

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$$= \frac{x}{2}\left[-\frac{1}{6}\int \cos 2x\, dx + \frac{1}{6}\int \cos x\, dx\right] + \frac{1}{4}\left[1 - \left(\frac{5}{4}D^2 + \frac{D^4}{4}\right) + \ldots\right]x$$

$$= \frac{x}{2}\left[-\frac{1}{12}\sin 2x + \frac{1}{6}\sin x\right] + \frac{1}{4}\left[1 - \frac{5}{4}D^2 + \ldots\right]x$$

$$= \frac{x}{12}\left[\sin x - \frac{1}{2}\sin 2x\right] + \frac{1}{4}[x - 0]$$

$$= \frac{x}{12}\left[\sin x - \frac{1}{2}\sin 2x\right] + \frac{x}{4}$$

Hence, the complete solution is

$$y = c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x + \frac{x}{12}\left(\sin x - \frac{1}{2}\sin 2x\right) + \frac{x}{4}.$$

EXERCISE 4.4

***Solve the following differential equations* [Q. 1 – 6] :**

1. (*i*) $\frac{d^3y}{dx^3} - 13\frac{dy}{dx} + 12y = x.$ (*ii*) $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = x$

(*iii*) $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 2y = 5 + 2x$ [*M.D.U. 2018*]

2. (*i*) $(D^3 - D^2 - 6D)y = x^2$ (*ii*) $(D^4 - 2D^3 + D^2)y = x^3$ (*iii*) $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 1 +$ ...

3. (*i*) $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + x^2 + x$ (*ii*) $(D^4 + 2D^3 - 3D^2)y = x^2 + 3e^{2x} + 4\sin x$ [*K.U. 201*

(*iii*) $(D^4 - a^4)y = x^4 + \sin bx$

4. (*i*) $(D-1)^2(D+1)^2y = \sin^2\frac{x}{2} + e^x + x$ (*ii*) $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 2e^{-x} + x^2$

5. $\frac{d^2y}{dx^2} + 4y = e^x + \sin 3x + x^2$ [*K.U. 2018*]

6. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 8(x^2 + e^{2x} + \sin 2x)$ [*M.D.U. 200*

7. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 2y = x - \sin x$ [*M.D.U. 2007*]

ANSWERS

1. (*i*) $y = c_1e^x + c_2e^{3x} + c_3e^{-4x} + \frac{1}{144}(12x + 13).$ (*ii*) $y = c_1e^x + c_2e^{-2x} + c_3e^{4x} + \frac{1}{8}\left(x + \frac{3}{4}\right).$

(*iii*) $y = c_1e^{-x/2} + c_2e^{-2x} + x$

2. (*i*) $y = c_1 + c_2e^{3x} + c_3e^{-2x} - \frac{1}{18}\left(x^3 - \frac{1}{2}x^2 + \frac{7}{6}x\right)$

(ii) $y = c_1 + c_2 x + (c_3 + c_4 x)e^x + \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2$.

(iii) $y = c_1 + c_2 e^{3x} + c_3 e^{-2x} - \frac{1}{108}(6x^3 - 3x^2 + 25x)$.

3. (i) $y = c_1 + (c_2 + c_3 x)e^{-x} + \frac{1}{18}e^{2x} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x$.

(ii) $y = (c_1 + c_2 x) + c_3 e^x + c_4 e^{-3x} - \left(\frac{1}{36}x^4 + \frac{2}{27}x^3 + \frac{7}{27}x^2\right) + \frac{3}{20}e^{2x} + \frac{2}{5}(\cos x + 2\sin x)$.

(iii) $y = c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax - \frac{1}{a^4}\left(x^4 + \frac{24}{a^4}\right) + \frac{\sin bx}{b^4 - a^4}$.

4. (i) $y = e^x(c_1 + c_2 x) + e^{-x}(c_3 + c_4 x) + \frac{1}{2}(1 + 2x) - \frac{1}{8}\cos x + \frac{x^2 e^x}{8}$.

(ii) $y = c_1 e^{-x} + c_2 \cos x + c_3 \sin x + xe^{-x} + (x^2 - 2x)$.

5. $y = c_1 \cos 2x + c_2 \sin 2x + \frac{e^x}{5} - \frac{\sin 3x}{5} + \frac{x^2}{4} - \frac{1}{8}$.

6. $y = e^{2x}(c_1 + c_2 x) + 2x^2 + 4x + 3 + 4x^2 e^{2x} + \cos 2x$.

7. $y = c_1 e^{(-2+\sqrt{2})x} + c_2 e^{(-2-\sqrt{2})x} + \frac{1}{2}(x - 2) + \frac{1}{17}(4\cos x - \sin x)$.

4.12.4. Case IV. To prove that $\frac{1}{f(D)}(e^{ax}V) = e^{ax}\frac{1}{f(D+a)}V$, where V is a function of x.

[M.D.U. 2014, 2000]

Proof. By successive differentiation, we have

$$D(e^{ax}V) = e^{ax}DV + ae^{ax}V = e^{ax}(D + a)V$$

$$D^2(e^{ax}V) = e^{ax}D^2V + ae^{ax}DV + a^2e^{ax}V + ae^{ax}DV$$

$$= e^{ax}(D^2 + 2aD + a^2)V = e^{ax}(D + a)^2V$$

Similarly, $$D^3(e^{ax}V) = e^{ax}(D + a)^3V$$

....

....

$$D^n(e^{ax}V) = e^{ax}(D + a)^nV$$

$\therefore$ $$f(D)(e^{ax}V) = e^{ax}f(D + a)V \quad ...(1)$$

Putting $f(D + a)V = X$, *i.e.*, $V = \frac{1}{f(D+a)}X$ in (1), we have

$$f(D)\left[e^{ax}.\frac{1}{f(D+a)}.X\right] = e^{ax}.X$$

Operating on both sides by $\frac{1}{f(D)}$, we have

$$\frac{1}{f(D)}.f(D)\left[e^{ax}.\frac{1}{f(D+a)}.X\right] = \frac{1}{f(D)}(e^{ax}X)$$

$$\Rightarrow \qquad e^{ax} \frac{1}{f(D+a)} X = \frac{1}{f(D)} (e^{ax} X)$$

$$\Rightarrow \qquad \frac{1}{f(D)} (e^{ax} X) = e^{ax} \frac{1}{f(D+a)} . X$$

In general, $$\frac{1}{f(D)} (e^{ax} V) = e^{ax} \frac{1}{f(D+a)} . V$$

Hence the required result.

Working Rule :

In general practice, take out e^{ax} and put $D + a$ for every D in $f(D)$, so that $f(D)$ becomes $f(D + a)$; then operate $\frac{1}{f(D+a)}$ with V alone by previous methods as discussed in case I and III.

Note.

Here V shall be of type $x^m \sin ax$ or $\cos ax$.

SOLVED EXAMPLES

Example 1.

Solve the differential equation $\frac{d^2y}{dx^2} + y = x . e^{2x}$

$\frac{1}{f(D)} e^{ax} . V$

V is any fun of x

$= \frac{e^{ax} . V}{f(D+a)}$

Solution. The given equation is $\frac{d^2y}{dx^2} + y = x . e^{2x}$

The symbolic form of the equation is $(D^2 + 1) y = x.e^{2x}$

Auxiliary equation is

$$D^2 + 1 = 0 \quad \Rightarrow \quad D = \pm i = 0 \pm i$$

$$\therefore \qquad \text{C.F.} = e^{0x} (c_1 \cos x + c_2 \sin x)$$

$$= c_1 \cos x + c_2 \sin x$$

Now, $$\text{P.I.} = \frac{1}{D^2 + 1} (xe^{2x}) = \frac{1}{D^2 + 1} (e^{2x} . x)$$

$$= e^{2x} \frac{1}{(D+2)^2+1} \cdot x$$ [*Note this step*]

$$= e^{2x} \cdot \frac{1}{D^2+4D+4+1} \cdot x = e^{2x} \cdot \frac{1}{D^2+4D+5} \cdot x$$

$$= e^{2x} \cdot \frac{1}{5\left(1+\frac{4D}{5}+\frac{D^2}{5}\right)} \cdot x$$

$$= \frac{1}{5} \cdot e^{2x} \left[1+\left(\frac{4D}{5}+\frac{D^2}{5}\right)\right]^{-1} \cdot x$$

$$= \frac{1}{5} e^{2x} \left[1-\left(\frac{4D}{5}+\frac{D^2}{5}\right)+\ldots\ldots\right] x$$

$$= \frac{1}{5} \cdot e^{2x} \left[x-\frac{4}{5} D(x)\right] = \frac{1}{5} e^{2x} \left(x-\frac{4}{5}\right).$$

Hence, the complete solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{5} e^{2x} \left(x - \frac{4}{5}\right).$$

Example 2.

Solve the differential equation $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = x^2 \cdot e^x$. [K.U. 2016, 15, 08, 06]

Solution. The given equation is $\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + 2y = x^2 \cdot e^x$

The symbolic form of the equation is

$$(D^3 - 3D + 2)\,y = x^2.e^x$$

Auxiliary equation is

$$D^3 - 3D + 2 = 0$$

or $$(D-1)(D^2+D-2) = 0$$

or $$(D-1)(D+2)(D-1) = 0$$

or $$(D-1)^2(D+2) = 0 \Rightarrow D = 1, 1, -2$$

$\therefore$ $$\text{C.F.} = (c_1 + c_2 x)\, e^x + c_3 e^{-2x}$$

Now, $\quad$ P.I. $= \dfrac{1}{D^3 - 3D + 2}(x^2 . e^x) = \dfrac{1}{D^3 - 3D + 2}(e^x . x^2)$

$$= e^x . \frac{1}{(D+1)^3 - 3(D+1) + 2} . x^2 = e^x . \frac{1}{D^3 + 3D^2} . x^2$$

e^x $\dfrac{1}{D^3+1 \quad +3D^2+}$

$3D \cdot 3D - \cancel{3} + 2$

$$= e^x . \frac{1}{3D^2\left(1 + \frac{D}{3}\right)} . x^2$$

$$= \frac{1}{3} e^x \frac{1}{D^2}\left(1 + \frac{D}{3}\right)^{-1} . x^2$$

x की Value नि open करते है bino mial

$$= \frac{1}{3} e^x \frac{1}{D^2}\left[1 - \frac{D}{3} + \frac{D^2}{9} .....\right] x^2$$ [By Binomial theorem]

$$= \frac{1}{3} e^x \frac{1}{D^2}\left[x^2 - \frac{1}{3} D(x^2) + \frac{1}{9} D^2(x^2)\right]$$

$$= \frac{1}{3} e^x \left[\frac{1}{D^2} x^2 - \frac{1}{3} . \frac{1}{D}(x^2) + \frac{1}{9} x^2\right]$$

$$= \frac{1}{3} e^x \left[\frac{1}{D} \int x^2\, dx - \frac{1}{3} \int x^2\, dx + \frac{1}{9} x^2\right]$$

$$= \frac{1}{3} e^x \left[\frac{1}{D}\left(\frac{x^3}{3}\right) - \frac{x^3}{9} + \frac{x^2}{9}\right]$$

$$= \frac{1}{3} e^x \left[\frac{1}{3} \int x^3\, dx - \frac{x^3}{9} + \frac{x^2}{9}\right] = \frac{1}{3} e^x \left[\frac{x^4}{12} - \frac{x^3}{9} + \frac{x^2}{9}\right]$$

$$= \frac{1}{108} x^2 e^x [3x^2 - 4x + 4]$$

Hence, the complete solution is

$$y = (c_1 + c_2 x)e^x + c_3 e^{-2x} + \frac{1}{108} x^2 e^x (3x^2 - 4x + 4).$$

Example 3.

Solve the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^x \sin 2x. \qquad \textit{[M.D.U. 2018]}$$

Solution. The given equation is $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = e^x \sin 2x$

The symbolic form of the equation is

$$(D^2 + 2D + 1)\,y = e^x \sin 2x$$

Auxiliary equation is $D^2 + 2D + 1 = 0$

$$(D+1)^2 = 0 \Rightarrow D = -1, -1$$

$\therefore$ C.F. $= (c_1 + c_2 x)\,e^{-x}$

Now, P.I. $= \dfrac{1}{D^2 + 2D + 1} \cdot e^x \sin 2x$

$$= \frac{1}{(D+1)^2}\, e^x \sin 2x = e^x \cdot \frac{1}{(D+1+1)^2} \sin 2x$$

$$= e^x \cdot \frac{1}{(D+2)^2} \sin 2x = e^x \cdot \frac{1}{D^2 + 4D + 4} \sin 2x$$

$$= e^x \cdot \frac{1}{-2^2 + 4D + 4} \cdot \sin 2x$$

$$= e^x \cdot \frac{1}{4D} \cdot \sin 2x = \frac{1}{4} e^x \int \sin 2x \, dx$$

$$= \frac{1}{4} e^x \cdot \frac{-\cos 2x}{2} = -\frac{1}{8} e^x \cos 2x$$

Hence, the complete solution is $y = (c_1 + c_2 x)\,e^{-x} - \dfrac{1}{8} e^x \cos 2x.$

Example 4.

Solve the differential equation

$$\frac{d^2 y}{dx^2} + 2y = x^2 \cdot e^{3x} + e^x \cdot \cos 2x \qquad \text{[M.D.U. 2015, 13, 11; K.U. 2007]}$$

Solution. The given equation is $\dfrac{d^2 y}{dx^2} + 2y = x^2 \cdot e^{3x} + e^x \cdot \cos 2x$

The symbolic form of the equation is

$$(D^2 + 2)\,y = x^2 e^{3x} + e^x \cos 2x$$

Auxiliary equation is $D^2 + 2 = 0$

$\Rightarrow$

$$D = \pm i\sqrt{2} = 0 \pm i\sqrt{2}$$

$\therefore$ C.F. $= e^{0x}(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x) = c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x$

Now, P.I. $= \dfrac{1}{D^2 + 2}(x^2 e^{3x} + e^x \cos 2x)$

$$= \frac{1}{D^2+2}(e^{3x}x^2) + \frac{1}{D^2+2}(e^x \cos 2x)$$

$$= e^{3x} \cdot \frac{1}{(D+3)^2+2} \cdot x^2 + e^x \cdot \frac{1}{(D+1)^2+2} \cos 2x$$

$$= e^{3x} \cdot \frac{1}{D^2+6D+11} x^2 + e^x \cdot \frac{1}{D^2+2D+3} \cos 2x$$

$$= e^{3x} \frac{1}{11\left(1+\frac{6}{11}D+\frac{D^2}{11}\right)} \cdot x^2 + e^x \frac{1}{-4+2D+3} \cos 2x$$

$$= \frac{1}{11} e^{3x} \left[1+\frac{6}{11}D+\frac{D^2}{11}\right]^{-1} \cdot x^2 + e^x \frac{1}{2D-1} \cos 2x$$

$$= \frac{1}{11} e^{3x} \left[1-\left(\frac{6}{11}D+\frac{D^2}{11}\right)+\left(\frac{6}{11}D+\frac{D^2}{11}\right)^2 \cdots\cdots\right] x^2 + e^x \cdot \frac{2D+1}{(2D-1)(2D+1)} \cdot \cos 2x$$

$$= \frac{1}{11} e^{3x} \left[1-\frac{6}{11}D-\frac{D^2}{11}+\frac{36}{121}D^2+\cdots\cdots\right] x^2 + e^x \cdot \frac{2D+1}{4D^2-1} \cdot \cos 2x$$

$$= \frac{1}{11} e^{3x} \left[1-\frac{6}{11}D+\frac{25}{121}D^2+\cdots\cdots\right] x^2 + e^x \cdot \frac{(2D+1)}{4(-4)-1} \cdot \cos 2x$$

$$= \frac{1}{11} e^{3x} \left[x^2-\frac{6}{11}D(x^2)+\frac{25}{121}D^2(x^2)\right] - \frac{e^x}{17}(2D+1)\cos 2x$$

$$= \frac{1}{11} e^{3x} \left[x^2-\frac{6}{11}.2x+\frac{25}{121}.2\right] - \frac{e^x}{17}[2D(\cos 2x)+\cos 2x]$$

$$= \frac{1}{11} e^{3x} \left[x^2-\frac{12}{11}x+\frac{50}{121}\right] - \frac{1.e^x}{17}[2(-2\sin 2x)+\cos 2x]$$

$$= \frac{1}{11} e^{3x} \left[x^2-\frac{12}{11}x+\frac{50}{121}\right] - \frac{1}{17}e^x[-4\sin 2x+\cos 2x]$$

$$= \frac{1}{11} e^{3x} \left[x^2-\frac{12}{11}x+\frac{50}{121}\right] + \frac{1}{17}e^x[4\sin 2x-\cos 2x]$$

Hence, the complete solution is

$$y = c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x + \frac{1}{11}e^{3x}\left(x^2-\frac{12}{11}x+\frac{50}{121}\right) + \frac{1}{17}e^x(4\sin 2x-\cos 2x).$$

EXERCISE 4.5

***Solve the following differential equations* [Q. 1 – 6] :**

1. (i) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x^2 . e^{3x}$ [M.D.U. 2012; K.U. 2000]

   (ii) $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} - 6y = x^2e^{2x}$

   (iii) $(D^2 - 5D + 6)\, y = x\, e^{4x}$.

2. (i) $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = x\,(x + e^x)$

   (ii) $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} - 6y = e^{2x}\,(1 + x)$.

3. (i) $\dfrac{d^3y}{dx^3} - 2\dfrac{dy}{dx} + 4y = e^x \cos x$

   (ii) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 4y = e^x \cos x$

   (iii) $\dfrac{d^4y}{dx^4} - y = e^x \cos x$

   (iv) $(D^2 + 3D + 2)\, y = e^{2x} \sin x$.

4. (i) $(D^2 - 4D + 3)\, y = e^x \cos 2x + \cos 3x$

   (ii) $(D^2 - 4D + 4)\, y = e^{2x} \cos^2 x$. [M.D.U. 2001]

5. (i) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = e^x \sin x$

   (ii) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = e^x \sin 2x + x^2$ [M.D.U. 2008]

6. $\dfrac{d^3y}{dx^3} - 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = e^{2x} \cos x + k$, where k is a constant.

ANSWERS

1. (i) $y = (c_1 + c_2x)\, e^x + \dfrac{e^{3x}}{8}(2x^2 - 4x + 3)$

   (ii) $y = c_1e^{-x} + c_2e^{3x} + c_3e^{-2x} - \dfrac{e^{2x}}{12}\left(x^2 + \dfrac{5}{6}x + \dfrac{97}{72}\right)$

   (iii) $y = c_1e^{2x} + c_2e^{3x} + \dfrac{1}{2}e^{4x}\left(x - \dfrac{3}{2}\right)$.

2. (i) $y = c_1e^{2x} + c_2e^{3x} + \dfrac{1}{6}\left(x^2 + \dfrac{5}{3}x + \dfrac{19}{18}\right) + \dfrac{1}{2}e^x\left(x + \dfrac{3}{2}\right)$

   (ii) $y = c_1e^{-x} + c_2e^{3x} + c_3e^{-2x} - \dfrac{1}{12}e^{2x}\left[x + \dfrac{17}{12}\right]$

3. (i) $y = c_1 e^{-2x} + e^x (c_2 \cos x + c_3 \sin x) + \frac{1}{20} xe^x (3 \sin x - \cos x)$

(ii) $y = e^x (c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x) + \frac{1}{2} e^x \cos x$

(iii) $y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \frac{1}{5} e^x \cos x$

(iv) $y = c_1 e^{-x} + c_2 e^{-2x} - \frac{1}{170} e^{2x} (7 \cos x - 11 \sin x)$.

4. (i) $y = c_1 e^x + c_2 e^{3x} - \frac{1}{8} e^x (\sin 2x + \cos 2x) - \frac{1}{30} (2 \sin 3x + \cos 3x)$

(ii) $y = (c_1 + c_2 x) e^{2x} + \frac{1}{8} e^{2x} (2x^2 - \cos 2x)$.

5. (i) $y = e^x (c_1 \cos x + c_2 \sin x) - \frac{x}{2} e^x \cos x$.

(ii) $y = (c_1 + c_2 x) e^{-x} - \frac{1}{8} e^x \cos 2x + (x^2 - 4x + 6)$.

6. $y = c_1 e^{2x} + c_2 \cos x + c_3 \sin x - \frac{e^{2x}}{8} (\cos x - \sin x) - \frac{k}{2}$.

4.12.5. Case V. To evaluate $\frac{1}{f(D)} \cdot (x V)$, where V is any function of x.

i.e., **Show that** $\frac{1}{f(D)} (x V) = x \cdot \frac{1}{f(D)} . V + \frac{d}{dD} \left[\frac{1}{f(D)} \right] . V$

Proof. By Leibnitz's Theorem on successive differentiation, we have

$$D^n (xV) = D^n (Vx) = (D^n V) x + {}^nC_1 (D^{n-1} V)$$
$$= x . D^n V + n . D^{n-1} V$$
$$= x D^n V + \frac{d}{dD} (D^n) V \qquad \left[\because \frac{d}{dD} D^n = nD^{n-1} \right]$$

$$\therefore \quad f(D)(xV) = x f(D) V + f'(D) V \qquad ...(1)$$

$$[\because f(D) = P_0 + P_1 D + P_2 D^2 + ... + P_n D^n]$$

Putting $f(D) V = X$, we have

$$\frac{1}{f(D)} [f(D) V] = \frac{1}{f(D)} X \quad \Rightarrow \quad V = \frac{1}{f(D)} . X$$

Also since V is a function of x, so is X

$\therefore$ From (1), $$f(\mathrm{D})\left[x\,\frac{1}{f(\mathrm{D})}\,.\,\mathrm{X}\right] = x\,.\,\mathrm{X} + f'(\mathrm{D})\,.\,\frac{1}{f(\mathrm{D})}\,\mathrm{X}$$

Operating on both sides by $\frac{1}{f(\mathrm{D})}$, we have

$$x\,\frac{1}{f(\mathrm{D})}\,\mathrm{X} = \frac{1}{f(\mathrm{D})}(x\,\mathrm{X}) + \frac{f'(\mathrm{D})}{[f(\mathrm{D})]^2}\,.\,\mathrm{X}$$

or $$\frac{1}{f(\mathrm{D})}(x\,\mathrm{X}) = x\,.\,\frac{1}{f(\mathrm{D})}\,.\,\mathrm{X} - \frac{f'(\mathrm{D})}{[f(\mathrm{D})]^2}\,.\,\mathrm{X}$$

or $$\frac{1}{f(\mathrm{D})}(x\,\mathrm{X}) = x\,.\,\frac{1}{f(\mathrm{D})}\,\mathrm{X} + \frac{d}{d\mathrm{D}}\left[\frac{1}{f(\mathrm{D})}\right].\,\mathrm{X} \qquad \left[\because \frac{d}{dD}\left(\frac{1}{f(D)}\right) = -\frac{f'(D)}{[f(D)]^2}\right]$$

In general, replacing X by V, we have

$$\frac{1}{f(\mathrm{D})}(x\mathrm{V}) = x\,.\,\frac{1}{f(\mathrm{D})}\,.\,\mathrm{V} + \frac{d}{d\mathrm{D}}\left[\frac{1}{f(\mathrm{D})}\right].\,\mathrm{V}$$

[$\because$ *X and V are both functions of x*]

Hence the result.

SOLVED EXAMPLES

Example 1.

Solve the differential equation $\frac{d^2y}{dx^2} + 4y = x \sin x$ *[M.D.U. 2015]*

$\frac{1}{f(D)}x.V = \frac{1}{f(D)}V + \frac{d}{d(D)}\frac{1}{f(D)}$

Solution. The given equation is $\frac{d^2y}{dx^2} + 4y = x \sin x$.

The symbolic form of the equation is

$$(\mathrm{D}^2 + 4)y = x \sin x$$

Auxiliary equation is $\mathrm{D}^2 + 4 = 0 \Rightarrow \mathrm{D} = \pm 2i = 0 \pm 2i$

$\therefore$ $$\text{C.F.} = e^{0x}(c_1 \cos 2x + c_2 \sin 2x)$$
$$= c_1 \cos 2x + c_2 \sin 2x$$

Now, $$\text{P.I.} = \frac{1}{\mathrm{D}^2 + 4}(x \sin x)$$

$$= x \cdot \frac{1}{D^2+4}\sin x - \frac{2D}{(D^2+4)^2}\sin x$$

$$\left[\because \frac{1}{f(D)}(xV) = x.\frac{1}{f(D)}V + \frac{d}{dD}\left(\frac{1}{f(D)}\right)\cdot V\right]$$

$$= x \cdot \frac{1}{-1+4}\sin x - \frac{2D}{(-1+4)^2}\cdot \sin x$$

$$= \frac{1}{3}x\sin x - \frac{2}{9}D(\sin x)$$

$$= \frac{1}{3}x\sin x - \frac{2}{9}\cdot\frac{d}{dx}(\sin x) = \frac{1}{3}x\sin x - \frac{2}{9}\cos x$$

Hence, the complete solution is

$$y = c_1\cos 2x + c_2\sin 2x + \frac{1}{3}x\sin x - \frac{2}{9}\cos x.$$

Example 2.

Solve the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x.$$ *[M.D.U. 2019, 17, 06; K.U. 201*

Solution. The given equation is $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x\sin x$

The symbolic form of the given equation is

$$(D^2 - 2D + 1)y = x e^x \sin x$$

Auxiliary equation is

$$D^2 - 2D + 1 = 0$$

or $$(D-1)^2 = 0 \Rightarrow D = 1, 1$$

$\therefore$ $$\text{C.F.} = (c_1 + c_2 x)e^x$$

Now, $$\text{P.I.} = \frac{1}{D^2 - 2D + 1}(xe^x\sin x) = \frac{1}{(D-1)^2}(xe^x\sin x)$$

$$= e^x \cdot \frac{1}{(D+1-1)^2}(x\sin x) = e^x \cdot \frac{1}{D^2}(x\sin x)$$

$$= e^x\left[x \cdot \frac{1}{D^2}\sin x + \frac{d}{dD}\left(\frac{1}{D^2}\right)\sin x\right]$$

$$= e^x\left[x \cdot \frac{1}{D^2}\sin x - \frac{2}{D^3}\cdot \sin x\right]$$

$$= e^x\left[x \cdot \frac{1}{-1}\sin x - \frac{2}{D}\left(\frac{1}{D^2}\cdot \sin x\right)\right]$$

$$= e^x\left[-x\sin x - \frac{2}{D}\cdot\frac{1}{-1}\sin x\right]$$

$$= e^x\left[-x\sin x + \frac{2}{D}\sin x\right] = e^x\left[-x\sin x + 2\int \sin x\,dx\right]$$

$$= e^x[-x\sin x + 2(-\cos x)] = -e^x[x\sin x + 2\cos x]$$

Hence, the complete solution is $y = (c_1 + c_2 x)e^x - e^x(x\sin x + 2\cos x)$.

Example 3.

Solve the differential equation

$$\frac{d^2y}{dx^2} - y = x^2\cos x. \qquad \text{[K.U. 2015, 11; M.D.U. 2014, 04]}$$

Solution. The given differential equation is

$$\frac{d^2y}{dx^2} - y = x^2\cos x.$$

The symbolic form of the equation is $(D^2 - 1)y = x^2\cos x$

Auxiliary equation is $D^2 - 1 = 0 \Rightarrow D = \pm 1$

$\therefore$ C.F. $= c_1e^x + c_2e^{-x}$

Now, P.I. $= \dfrac{1}{D^2-1}(x^2\cos x)$

$$= \text{Real part of } \frac{1}{D^2-1}(x^2 e^{ix}) \qquad [\because e^{ix} = \cos x + i\sin x]$$

$$= \text{Real part of } e^{ix}\frac{1}{(D+i)^2-1}x^2 \qquad \left[\because \frac{1}{f(D)}Xe^{ax} = e^{ax}\cdot\frac{1}{f(D+a)}X\right]$$

$$= \text{R.P. of } e^{ix}\frac{1}{D^2+2iD-2}x^2$$

$$= \text{R.P. of } e^{ix}\frac{1}{-2\left(1 - iD - \dfrac{D^2}{2}\right)}x^2$$

$$= \text{R.P. of } \frac{-e^{ix}}{2}\left[1-\left(iD+\frac{D^2}{2}\right)\right]^{-1} x^2$$

$$= \text{R.P. of } \frac{-e^{ix}}{2}\left[1+\left(iD+\frac{D^2}{2}\right)+\left(iD+\frac{D^2}{2}\right)^2+\ldots\ldots\right]x^2.$$

$$= \text{R.P. of } -\frac{1}{2}e^{ix}\left[1+iD+\frac{D^2}{2}+i^2D^2+\ldots\ldots\right]x^2$$

$$= \text{R.P. of } -\frac{1}{2}e^{ix}\left[1+iD-\frac{D^2}{2}+\ldots\ldots\right]x^2$$

$$= \text{R.P. of } -\frac{1}{2}e^{ix}\left[x^2+iDx^2-\frac{1}{2}D^2x^2+\ldots\ldots\right]$$

$$= \text{R.P. of } -\frac{1}{2}(\cos x+i\sin x)(x^2+2ix-1)$$

$$= -\frac{1}{2}x^2\cos x+\frac{1}{2}\cos x+x\sin x$$

$$= -\frac{1}{2}(x^2-1)\cos x+x\sin x$$

Hence, the complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

i.e.,
$$y = c_1e^x+c_2e^{-x}-\frac{1}{2}(x^2-1)\cos x+x\sin x$$

Example 4.

Solve the differential equation $\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+4y=3x^2e^{2x}\sin 2x.$

Solution. The given differential equation is

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=3x^2e^{2x}\sin 2x$$

The symbolic form of the equation is

$$(D^2-4D+4)y=3x^2e^{2x}\sin 2x$$

Auxiliary equation is

$$D^2-4D+4=0$$

or
$$(D-2)^2=0 \Rightarrow D=2,2$$

$$\text{C.F.} = (c_1 + c_2 x)\, e^{2x}$$

Now, $$\text{P.I.} = \frac{1}{(D-2)^2}\,(3x^2 e^{2x} \sin 2x)$$

$$= 3e^{2x} \cdot \frac{1}{(D+2-2)^2}\,(x^2 \sin 2x) = 3e^{2x}\,\frac{1}{D^2}\,(x^2 \sin 2x)$$

$$= \text{I.P. of } 3e^{2x} \cdot \frac{1}{D^2}\, x^2 e^{i2x} \qquad [\because e^{i2x} = \cos 2x + i \sin 2x]$$

$$= \text{I.P. of } 3e^{2x} \cdot e^{i2x}\, \frac{1}{(D+2i)^2}\, x^2$$

$$= \text{I.P. of } 3e^{2x} \cdot e^{i2x}\, \frac{1}{\left[2i\left(1+\frac{D}{2i}\right)\right]^2}\,(x^2)$$

$$= \text{I.P. of } 3e^{2x} \cdot e^{i2x} \cdot \frac{-1}{4}\left(1 - \frac{iD}{2}\right)^{-2} (x^2) \qquad [\because i^2 = -1]$$

$$= \text{I.P. of } \frac{-3}{4}\, e^{2x} \cdot e^{i2x} \left[1 + 2 \cdot \frac{iD}{2} + 3\left(\frac{iD}{2}\right)^2 + \ldots\right] x^2$$

$$= \text{I.P. of } \frac{-3}{4}\, e^{2x} \cdot e^{i2x} \left[1 + iD - \frac{3}{4} D^2 + \ldots\right] x^2 \qquad \textit{[By binomial theorem]}$$

$$= \text{I.P. of } \frac{-3}{4}\, e^{2x} \cdot e^{i2x} \left[x^2 + 2ix - \frac{3}{2}\right]$$

$$= \text{I.P. of } \frac{-3}{4}\, e^{2x} (\cos 2x + i \sin 2x)\left(x^2 + 2ix - \frac{3}{2}\right)$$

$$= \frac{-3}{4}\, e^{2x} \left[2x \cos 2x + \left(x^2 - \frac{3}{2}\right) \sin 2x\right]$$

$$= \frac{-3}{8}\, e^{2x} \left[4x \cos 2x + (2x^2 - 3) \sin 2x\right]$$

Hence, the complete solution is given by

$$y = \text{C.F.} + \text{P.I.}$$

$$y = (c_1 + c_2 x)\, e^{2x} - \frac{3}{8}\, e^{2x}\, [4x \cos 2x + (2x^2 - 3) \sin 2x].$$

EXERCISE 4.6

Solve the following differential equations [Q. 1 – 8]:

1. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x \sin x.$ [M.D.U. 2008]

2. $\frac{d^2y}{dx^2} + 4y = x \cos x.$

3. $\frac{d^2y}{dx^2} - 4y = x \cos 2x.$

4. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = xe^x \sin x$ [K.U. 2011, 09, 0

5. (i) $\frac{d^4y}{dx^4} - y = x \sin x$ [M.D.U. 2003] (ii) $\frac{d^4y}{dx^4} - y = x \sin x + e^x.$ [M.D.U. 20

$\left[\textbf{Hint.} \text{ (i) P.I.} = \frac{1}{D^4 - 1}(x \sin x) = \text{Imaginary part of } \frac{1}{D^4 - 1} xe^{ix} = \text{I.P. of } e^{ix} \frac{1}{(D+i)^4 - 1} x.\right]$

6. $\frac{d^2y}{dx^2} - y = x \sin x + (1 + x^2)\, e^x.$

7. $\frac{d^2y}{dx^2} - y = x \sin x + x^2\, e^x.$

8. $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = x^2 \sin x.$ [M.D.U. 2015]

ANSWERS

1. $y = (c_1 + c_2 x)\, e^x + \frac{x}{2} \cos x + \frac{1}{2}(\cos x - \sin x)$

2. $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{3} x \cos x + \frac{2}{9} \sin x$

3. $y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{8} x \cos 2x + \frac{1}{16} \sin 2x.$

4. $y = c_1 e^{-x} + c_2 e^{-2x} + e^x \left[\frac{x}{10}(\sin x - \cos x) + \frac{1}{10} \cos x - \frac{1}{25} \sin x\right]$

5. (i) $y = c_1 \cos x + c_2 \sin x + c_3 e^x + c_4 e^{-x} + \frac{1}{8} x^2 \cos x - \frac{3}{8} x \sin x$

   (ii) $y = c_1 \cos x + c_2 \sin x + c_3\, e^x + c_4 e^{-x} + \frac{1}{8} x^2 \cos x - \frac{3}{8} x \sin x + \frac{xe^x}{4}$

6. $y = c_1 e^x + c_2\, e^{-x} - \frac{1}{2}(x \sin x + \cos x) + \frac{1}{12} xe^x (2x^2 - 3x + 9).$

7. $y = c_1 e^x + c_2\, e^{-x} - \frac{1}{2}(x \sin x + \cos x) + \frac{1}{12} xe^x\, (2x^2 - 3x + 3).$

8. $y = (c_1 + c_2\, x) \cos x + (c_3 + c_4 x) \sin x - \frac{x^3}{12} \cos x - \left(\frac{x^4}{48} - \frac{3}{16} x^2\right) \sin x.$

5 HOMOGENEOUS LINEAR EQUATIONS

5.1. HOMOGENEOUS LINEAR EQUATION

Definition. *An equation of the form*

$$P_0 x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots.. + P_n y = X \qquad \ldots(1)$$

where $P_0, P_1, P_2, \ldots, P_n$ are constants and X is a function of x, is called a homogeneous linear differential equation.

5.2. METHOD OF SOLUTION

To reduce the homogeneous linear equation

$$\mathbf{P_0 x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots.. + P_n y = X}$$

into linear equation with constant coefficients.

Also prove that

$$\mathbf{x^n \frac{d^n y}{dx^n} = D (D-1)(D-2) \ldots\ldots (D-n+1)\, y.}$$

Proof. The given homogeneous equation is

$$P_0 x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots.. + P_{n-1} x \frac{dy}{dx} + P_n y = X \qquad \ldots(1)$$

Put $x = e^z$ so that $z = \log x$ and $\frac{dz}{dx} = \frac{1}{x}$ $\qquad \ldots(2)$

Now, $$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x} \qquad [\textit{Using } (2)]$$

SOLVED EXAMPLES

Example 1.

Solve the differential equation

$$x^2\frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x}$$ [K.U. 2019, 06; M.D.U. 2017]

Solution. The given equation is

$$x^2\frac{d^2y}{dx^2} - 2y = x^2 + \frac{1}{x} \quad ...(1)$$

Put $x = e^z$ so that $z = \log x$ and $D = x\frac{d}{dx} = \frac{d}{dz}$. Then, given equation (1) becomes

$$[D(D-1)-2]\,y = e^{2z} + e^{-z}$$

$$[D^2 - D - 2]\,y = e^{2z} + e^{-z}$$

$x^2\frac{d^2}{dx^2} = [D(D-1)]$

Auxiliary equation is $D^2 - D - 2 = 0$

$$(D-2)(D+1) = 0 \quad i.e., \quad D = 2, -1$$

$\therefore$ $$\text{C.F.} = c_1 e^{2z} + c_2 e^{-z}$$

$$\text{P.I.} = \frac{1}{D^2 - D - 2}(e^{2z} + e^{-z})$$

$$= \frac{1}{D^2 - D - 2}e^{2z} + \frac{1}{D^2 - D - 2}e^{-z}$$

$$= z\,.\,\frac{1}{2D-1}e^{2z} + z\,.\,\frac{1}{2D-1}\,.\,e^{-z}$$ [Cases of failure]

$$= z\,.\,\frac{1}{2(2)-1}\,.\,e^{2z} + z\,\frac{1}{2(-1)-1}\,.\,e^{-z}$$

$$= \frac{1}{3}ze^{2z} - \frac{1}{3}ze^{-z} = \frac{1}{3}\,.\,z(e^{2z} - e^{-z})$$

Hence the complete solution is

$$y = c_1 e^{2z} + c_2 e^{-z} + \frac{1}{3}(e^{2z} - e^{-z})z$$

$$y = c_1 x^2 + \frac{c_2}{x} + \frac{1}{3}\left(x^2 - \frac{1}{x}\right)\log x.$$ $[\because x = e^z]$

Example 2.

Solve the differential equation $x\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}=\frac{1}{x}$.

Solution. The given equation is

$$x\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}=\frac{1}{x}$$

Multiplying both sides by x^2, we have

$$x^3\frac{d^3y}{dx^3}+x^2\frac{d^2y}{dx^2}=x$$ $D(D-1)(D-2)=x^3\frac{d^3}{dx^3}$

Put $x=e^z$ so that $z=\log x$

Let $x\frac{d}{dx}=\frac{d}{dz}=D$. Then the given equation (1), becomes

$$[D(D-1)(D-2)+D(D-1)]\,y=e^z$$ $[D(D^2-3D+2)+D(D-1)]$

or $$(D^3-2D^2+D)\,y=e^z$$ $D[D^2-3D+2+D-1]$

$\therefore$ Auxiliary equation is $D^3-2D^2+D=0$ $D[D^2-2D+1]=0$

or $$D(D-1)^2=0 \Rightarrow D=0,1,1$$ $D=0,\ D=1,1$

$\therefore$ $$\text{C.F.}=c_1e^{0z}+(c_2+c_3z)\,e^z=c_1+(c_2+c_3z)\,e^z$$

Now, $$\text{P.I.}=\frac{1}{D^3-2D^2+D}\cdot e^z$$ [Case of failu...

$$=z\cdot\frac{1}{3D^2-4D+1}\cdot e^z$$ [Case of failu...

$$=z^2\cdot\frac{1}{6D-4}\cdot e^z$$

$$=z^2\cdot\frac{1}{6(1)-4}\cdot e^z=\frac{z^2}{2}\cdot e^z$$

Hence the complete solution is

$$y=c_1+(c_2+c_3z)\,e^z+\frac{z^2}{2}\cdot e^z$$

or $$y=c_1+(c_2+c_3\log x)\,x+\frac{x}{2}(\log x)^2$$ [$\because e^z$...

Example 3.

Solve the differential equation $x^2\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}-4y=x^4.$

Solution. The given equation is $x^2\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}-4y=x^4$ ...(1)

Put $x=e^z$ so that $z=\log x$ and let $D=\dfrac{d}{dz}=x\dfrac{d}{dx}$

Then the given equation (1) becomes

$$[D(D-1)-2D-4]\,y=e^{4z}$$

or $$(D^2-3D-4)\,y=e^{4z}$$

$\therefore$ Auxiliary equation is $D^2-3D-4=0$

$$(D-4)(D+1)=0 \Rightarrow D=4,-1$$

$\therefore$ $$C.F.=c_1e^{4z}+c_2e^{-z}$$

$$=c_1x^4+\frac{c_2}{x} \qquad [\because x=e^z]$$

Now, $$P.I.=\frac{1}{D^2-3D-4}\cdot e^{4z}$$

$$=\frac{1}{16-12-4}\cdot e^{4z}=\frac{1}{0}e^{4z} \qquad [\textit{Case of failure}]$$

$$=z\cdot\frac{1}{2D-3}e^{4z}$$

[*Differentiating denominator w.r.t. D and multiplying by z*]

$$=z\cdot\frac{1}{2.4-3}\cdot e^{4z}$$

$$=\frac{z\cdot e^{4z}}{5}=\frac{1}{5}(\log x)\,x^4 \qquad [\because e^z=x]$$

Hence the complete or general solution is

$$y=c_1x^4+\frac{c_2}{x}+\frac{x^4.\log x}{5} \qquad [\because C.S.=C.F.+P.I.]$$

Example 4.

Solve the differential equation

$$(x^2D^2-3xD+4)\,y=x^m.$$

Solution. The given equation is $(x^2D^2-3xD+4)\,y=x^m$

Put $x = e^z$ so that $z = \log x$

and let $D' = x\frac{d}{dx} = \frac{d}{dz}$. Then the given equation becomes

$x^2D^2 = D'(D'-1)$

$$[D'(D'-1) - 3D' + 4]\,y = e^{mz}$$

or
$$[(D'^2 - 4D' + 4]\,y = e^{mz} \Rightarrow (D'-2)^2 y = e^{mz}$$

$\therefore$ Auxiliary equation is $(D'-2)^2 = 0 \Rightarrow D' = 2, 2$

$\therefore$
$$\text{C.F.} = (c_1 + c_2 z)\,e^{2z}$$
$$= (c_1 + c_2 \log x)\,x^2$$

Now,
$$\text{P.I.} = \frac{1}{(D'-2)^2}\cdot e^{mz} = \frac{1}{(m-2)^2}\,e^{mz} = \frac{1}{(m-2)^2}\cdot x^m$$

Hence the complete solution is

$$y = (c_1 + c_2 \log x)\,x^2 + \frac{1}{(m-2)^2}\cdot x^m$$

Example 5.

Solve the differential equation

$$x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right).$$

Solution. The given equation is

$$x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$$

Put $x = e^z$ so that $z = \log x$ and $D = x\frac{d}{dx} = \frac{d}{dz}$.

Then given equation (1) becomes

$$[D(D-1)(D-2) + 2D(D-1) + 2]y = 10\,(e^z + e^{-z})$$

or
$$(D^3 - D^2 + 2)\,y = 10\,(e^z + e^{-z})$$

Auxiliary equation is $D^3 - D^2 + 2 = 0$

or
$$(D+1)(D^2 - 2D + 2) = 0$$

$\therefore$
$$D = -1, \frac{2 \pm \sqrt{4-8}}{2} \Rightarrow D = -1, 1 \pm i$$

$\therefore$
$$\text{C.F.} = c_1e^{-z} + e^z\,(c_2 \cos z + c_3 \sin z)$$
$$= c_1x^{-1} + x\,[c_2 \cos(\log x) + c_3 \sin(\log x)]$$

Now,

$$\text{P.I.} = \frac{1}{D^3 - D^2 + 2} \cdot 10(e^z + e^{-z})$$

$$= 10 \cdot \frac{1}{D^3 - D^2 + 2} e^z + 10 \frac{1}{D^3 - D^2 + 2} e^{-z}$$

[Second term is a case of failure]

$$= 10 \cdot \frac{1}{1 - 1 + 2} \cdot e^z + 10 \cdot z \frac{1}{3D^2 - 2D} \cdot e^{-z}$$

[Multiplying second term by z and differentiating its den. w.r.t. D]

$$= 5e^z + 10z \cdot \frac{1}{3.1 - 2(-1)} \cdot e^{-z} = 5\, e^z + 2\, z e^{-z}$$

$$= \left(5x + 2 \log x \cdot \frac{1}{x}\right) \qquad [\because x = e^z]$$

Hence the complete solution is

$$y = c_1 x^{-1} + x\, [c_2 \cos(\log x) + c_3 \sin(\log x)] + 5x + \frac{2}{x} \log x.$$

Example 6.

Solve the differential equation

$$x^2 \frac{d^2 y}{dx^2} + 8x \frac{dy}{dx} + 13y = \log x \qquad \text{[K.U. 2007, 05, 2000]}$$

Solution. The given equation is

$$x^2 \frac{d^2 y}{dx^2} + 8x \frac{dy}{dx} + 13y = \log x$$ $\log e^z = z$

Put $x = e^z$ so that $z = \log x$ and $D = x\frac{d}{dx} = \frac{d}{dz}$. Then the given equation becomes

$$[D(D-1) + 8D + 13]\, y = z$$

$$(D^2 + 7D + 13)\, y = z$$

Auxiliary equation is $D^2 + 7D + 13 = 0$

$$\therefore \quad D = \frac{-7 \pm \sqrt{49 - 52}}{2} = -\frac{7}{2} \pm \frac{i\sqrt{3}}{2}$$

$$\therefore \quad \text{C.F.} = e^{-\frac{7}{2}z}\left(c_1 \cos \frac{\sqrt{3}}{2}.z + c_2 \sin \frac{\sqrt{3}}{2}.z\right)$$

or $$\text{C.F.} = x^{-7/2}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\log x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\log x\right)\right]$$ [∵ e ...

Now, $$\text{P.I.} = \frac{1}{D^2+7D+13}\cdot z = \frac{1}{13\left[1+\frac{7}{13}D+\frac{D^2}{13}\right]}\cdot z$$

$$= \frac{1}{13}\cdot\left[1+\frac{7}{13}D+\frac{D^2}{13}\right]^{-1}\cdot z$$

$$= \frac{1}{13}\left[1-\left(\frac{7}{13}D+\frac{D^2}{13}\right)+\ldots\ldots\right]\cdot z$$ [By Binomial theorem]

$$= \frac{1}{13}\left[1-\frac{7}{13}D\ldots\ldots\right]z = \frac{1}{13}\left[z-\frac{7}{13}D(z)\right]$$

$$= \frac{1}{13}\left[z-\frac{7}{13}\frac{d}{dz}(z)\right]$$ $\left[\because D = \frac{d}{dz}\right]$

$$= \frac{1}{13}\left(z-\frac{7}{13}\right) = \frac{1}{169}(13z-7)$$

z की कीमती क्या अगले हैं।

$$= \frac{1}{169}(13\log x - 7)$$ [∵ $e^z = x$]

Hence the complete the solution is

$$y = x^{-7/2}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\log x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\log x\right)\right] + \frac{1}{169}(13\log x - 7).$$

Example 7.

Solve the differential equation

$$x^2.\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y = \frac{1}{(1-x)^2}$$ *[M.D.U. 2013, 12; K.U. 201...]*

Solution. The given equation is

$$x^2.\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+y = \frac{1}{(1-x)^2}$$

Put $x = e^z$ so that $z = \log x$ and let $D = x\frac{d}{dx} = \frac{d}{dz}$. Then the given equation reduces to

$$[D(D-1)+3D+1]\,y = \frac{1}{(1-e^z)^2}$$

$$(D^2+2D+1)\,y = \frac{1}{(1-e^z)^2}$$

$\frac{1}{D-\alpha}x = e^{\alpha x}\int e^{-\alpha x}x\,dx$

Auxiliary equation is $D^2 + 2D + 1 = 0$

$$(D+1)^2 = 0 \quad \Rightarrow \quad D = -1, -1$$

$\therefore$ C.F. $= (c_1 + c_2 z)\,e^{-z} = (c_1 + c_2 \log x)\,\dfrac{1}{x}$ $\quad [\because e^z = x]$

Now, P.I. $= \dfrac{1}{D^2+2D+1}\cdot\dfrac{1}{(1-e^z)^2} = \dfrac{1}{(D+1)^2}\cdot\dfrac{1}{(1-e^z)^2}$

$$= \frac{1}{(D+1)(D+1)}\cdot\frac{1}{(1-e^z)^2} = \frac{1}{D+1}\left[\frac{1}{D+1}\cdot\frac{1}{(1-e^z)^2}\right]$$

$$= \frac{1}{D+1}\cdot e^{-z}\int\frac{1}{(1-e^z)^2}\cdot e^z\,dz \qquad \left[\because \frac{1}{D-a}X = e^{ax}\int X e^{-ax}\,dx\right]$$

$$= \frac{1}{D+1}\cdot e^{-z}\int(1-t)^{-2}\,dt, \text{ where } e^z = t$$

Put $1 - e^z = t$

$-e^z dx = dt$

$$= \frac{1}{D+1}\cdot e^{-z}\left[-\int(1-t)^{-2}\,(-dt)\right]$$

$$= \frac{1}{D+1}\cdot e^{-z}\,\frac{-(1-t)^{-1}}{-1} \qquad \left[\because \int[f(x)]^n\cdot f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$= \frac{1}{D+1}\cdot e^{-z}\cdot\frac{1}{1-t} = \frac{1}{D+1}\cdot e^{-z}\,\frac{1}{1-e^z} \qquad [\because t = e^z]$$

$$= \frac{1}{D+1}\cdot\frac{e^{-z}}{1-e^z}$$

$$= e^{-z}\int\frac{e^{-z}}{1-e^z}\cdot e^z\,dz \qquad \left[\because \frac{1}{D-a}X = e^{ax}\int Xe^{-ax}\,dx\right]$$

$$= e^{-z}\int\frac{dz}{1-e^z} = e^{-z}\int\frac{e^{-z}}{e^{-z}-1}\,dz \qquad [\textit{Multiplying num. and den. by } e^{-z}]$$

$e^{-z} - 1 = t$

$$= -e^{-z}\int\frac{-e^{-z}}{e^{-z}-1}\,dz$$

$$= -e^{-z}.\log(e^{-z}-1) \qquad \left[\because \int\frac{f'(z)}{f(z)}\,dz = \log[f(z)]\right]$$

$$= -\frac{1}{x}\log\left(\frac{1}{x}-1\right)$$

$$= -\frac{1}{x}\log\left(\frac{1-x}{x}\right) = \frac{1}{x}\log\left(\frac{1-x}{x}\right)^{-1}$$

$$= \frac{1}{x}\log\frac{x}{1-x}$$

Hence the complete solution is $y = (c_1 + c_2 \log x)\frac{1}{x} + \frac{1}{x}\log\frac{x}{1-x}$.

Example 8.

Solve the differential equation :

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x$$ [K.U. 2018; M.D.U. 2016, 0...]

Solution. The given equation is

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x$$

Put $x = e^z$ so that $z = \log x$

Let $D = x\frac{d}{dx}$. Then the given equation becomes

$$[D(D-1) - D - 3]\,y = z\,e^{2z}$$

or $$[D^2 - 2D - 3]\,y = z\,e^{2z}$$

Auxiliary equation is $D^2 - 2D - 3 = 0$

i.e., $$(D-3)(D+1) = 0 \Rightarrow D = 3, -1$$

$\therefore$ C.F. $= c_1 e^{3z} + c_2 e^{-z}$

Now, $$\text{P.I.} = \frac{1}{D^2 - 2D - 3}(ze^{2z})$$

$$= e^{2z}\frac{1}{(D+2)^2 - 2(D+2) - 3}\cdot z \qquad \left[\because \frac{1}{f(D)}(Xe^{ax}) = e^{ax}\frac{1}{f(D+a)}X\right]$$

$$= e^{2z}\frac{1}{D^2 + 2D - 3}z = e^{2z}\frac{1}{-3\left(1 - \frac{2D}{3} - \frac{D^2}{3}\right)}z$$

$$= \frac{e^{2z}}{-3}\left[1 - \left(\frac{2D}{3} + \frac{D^2}{3}\right)\right]^{-1} z$$

$$= -\frac{1}{3}e^{2z}\left[1+\frac{2D}{3}\right]z \qquad \text{[By Binomial theorem]}$$

$$= -\frac{1}{3}e^{2z}\left[z+\frac{2}{3}\right]$$

Hence the complete solution is

$$y = c_1e^{3z} + c_2e^{-z} - \frac{1}{3}e^{2z}\left(z+\frac{2}{3}\right)$$

or

$$y = c_1x^3 + c_2x^{-1} - \frac{1}{3}x^2\left(\log x+\frac{2}{3}\right). \qquad [\because e^z = x \Rightarrow z = \log x]$$

Example 9.

Solve the differential equation $(x^2D^2 - 3xD + 5)y = \sin(\log x)$. [M.D.U. 2019, 15]

Solution. The given equation is

$$(x^2D^2 - 3xD + 5)y = \sin(\log x)$$

Put $x = e^z$ so that $z = \log x$

Let $D' \equiv x D \equiv x\cdot\frac{d}{dx}$. Then the given equation becomes

$xD = D', \quad x^2D^2 = D'(D'-1)$

$$[D'(D'-1) - 3D' + 5]y = \sin z$$

or

$$(D'^2 - 4D' + 5)y = \sin z$$

Auxiliary equation is $D'^2 - 4D' + 5 = 0$

$$\therefore \quad D' = \frac{4\pm\sqrt{16-20}}{2} = 2\pm i$$

$$\therefore \quad \text{C.F.} = e^{2z}(c_1\cos z + c_2\sin z)$$

Now,

$$\text{P.I.} = \frac{1}{D'^2-4D'+5}(\sin z) = \frac{1}{-1-4D'+5}\sin z$$

$= \frac{1}{-4D+4}\sin z$

$\frac{1}{4}\frac{1}{(1-D)}\sin z =$

$$= \frac{1}{4}\cdot\frac{1}{1-D'}\times\frac{1+D'}{1+D'}\sin z = \frac{1}{4}\cdot\frac{(1+D')}{1-D'^2}\sin z$$

$$= \frac{1}{4}\cdot\frac{(1+D')}{[1-(-1)]}\sin z$$

$$= \frac{1}{8}[\sin z + D'(\sin z)] = \frac{1}{8}[\sin z + \cos z]$$

Hence the complete solution is y = C.F. + P.I.

i.e., $$y = e^{2z}(c_1 \cos z + c_2 \sin z) + \frac{1}{8}(\sin z + \cos z)$$

or $$y = x^2 [c_1 \cos(\log x) + c_2 \sin(\log x)] + \frac{1}{8}[\sin(\log x) + \cos(\log x)]$$

$[\because e^z = x \Rightarrow z = \log x]$

Example 10.

Solve $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = x^2 e^x.$ [M.D.U. 2018]

Solution. The given equation is

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2 e^x$$

Put $x = e^z$ so that $z = \log x$

Let $D \equiv x\dfrac{d}{dx} = \dfrac{d}{dz}$. Then the given equation becomes

$$[D(D-1) + D - 1]\, y = e^{2z} e^{e^z} \quad i.e., \quad (D^2 - 1)\, y = e^{2z} e^{e^z}$$

Auxiliary equation is

$$D^2 - 1 = 0 \Rightarrow D = \pm 1$$

$\therefore$ $$\text{C.F.} = c_1 e^z + c_2 e^{-z}$$

Now, $$\text{P.I.} = \frac{1}{D^2 - 1}(e^{2z} e^{e^z}) = \frac{1}{(D+1)(D-1)}(e^{2z} e^{e^z})$$

$$= \frac{1}{2}\left[\frac{1}{D-1} - \frac{1}{D+1}\right](e^{2z} e^{e^z})$$ [Using partial fractions]

$$= \frac{1}{2} \cdot \frac{1}{D-1}(e^{2z} e^{e^z}) - \frac{1}{2} \cdot \frac{1}{D+1}(e^{2z} e^{e^z})$$

$$= \frac{1}{2} e^z \int e^{2z} e^{e^z} e^{-z}\, dz - \frac{1}{2} e^{-z} \int e^{2z} e^{e^z} e^z\, dz$$

$$\left[\because \frac{1}{D-a} X = e^{ax} \int X e^{-ax}\, dx\right]$$

$$= \frac{1}{2} e^z \int e^t\, dt - \frac{1}{2} e^{-z} \int t^2 e^t\, dt, \quad \text{where } e^z = t$$

$$= \frac{1}{2} e^z e^t - \frac{1}{2} e^{-z} (t^2 - 2t + 2) e^t$$

$$= \frac{1}{2} e^z e^{e^z} - \frac{1}{2} e^{-z} (e^{2z} - 2e^z + 2) e^{e^z}$$

$$= \frac{1}{2} e^{e^z} [e^z - e^z + 2 - 2e^{-z}] = e^{e^z} - e^{-z} e^{e^z}$$

Hence the complete solution is y = C.F. + P.I.

i.e., $$y = c_1 e^z + c_2 e^{-z} + e^{e^z} - e^{-z} e^{e^z}$$

or $$y = c_1 x + c_2 x^{-1} + e^x (1 - x^{-1})$$ $[\because e^z = x]$

EXERCISE 5.1

Solve the following differential equations **[Q. 1 – 7] :**

1. (i) $(x^2 D^2 + 2xD - 2)\, y = 0$

(ii) $x^3 \dfrac{d^3y}{dx^3} + 6x^2 \dfrac{d^2y}{dx^2} + 4x \dfrac{dy}{dx} - 4y = 0$ [M.D.U. 2015, 07]

(iii) $x^2 \dfrac{d^2y}{dx^2} + 9x \dfrac{dy}{dx} + 25y = 50$

(iv) $\dfrac{d^3y}{dx^3} - \dfrac{4}{x} \cdot \dfrac{d^2y}{dx^2} + \dfrac{5}{x^2} \dfrac{dy}{dx} - \dfrac{2y}{x^3} = 1.$ [K.U. 2015]

2. (i) $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} - y = x^m$

(ii) $x^2 \dfrac{d^3y}{dx^3} - 4x \dfrac{d^2y}{dx^2} + 6 \dfrac{dy}{dx} = 4$

(iii) $x^4 \dfrac{d^3y}{dx^3} + 2x^3 \dfrac{d^2y}{dx^2} - x^2 \dfrac{dy}{dx} + xy = 1$

3. (i) $x^2 \dfrac{d^2y}{dx^2} - 2x \dfrac{dy}{dx} + 2y = 4x^3$

(ii) $x^3 \dfrac{d^3y}{dx^3} - x^2 \dfrac{d^2y}{dx^2} + 2x \dfrac{dy}{dx} - 2y = x^3 + 3x$

(iii) $x^2 \dfrac{d^2y}{dx^2} + 2x \dfrac{dy}{dx} - 20y = (x + 1)^2.$

4. (i) $x^2 \dfrac{d^2y}{dx^2} - x \dfrac{dy}{dx} + 2y = x \log x$

(ii) $x^2 \dfrac{d^2y}{dx^2} - x \dfrac{dy}{dx} + y = 2 \log x$ [K.U. 2001; M.D.U. 2001]

(iii) $\dfrac{d^2y}{dx^2} + \dfrac{1}{x} \dfrac{dy}{dx} = \dfrac{12 \log x}{x^2}.$

5. (i) $[x^4 D^4 + 6x^3 D^3 + 9x^2 D^2 + 3x D + 1]\, y = (1 + \log x)^2$

(ii) $[x^4 D^4 + 2x^3 D^3 + x^2 D^2 - x D + 1]\, y = x + \log x$

(iii) $x^2 \dfrac{d^2y}{dx^2} - 3x \dfrac{dy}{dx} + y = \dfrac{\log x \sin (\log x) + 1}{x}.$ [M.D.U. 2011]

6 LINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER

6.1. INTRODUCTION

The general form of linear differential equations of second order is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

where P, Q and R are functions of x only.

If P and Q are constants, then the equation can be solved by the methods discussed in the previous chapter. Otherwise there is no general method of solving these type of equations. However, we can find solution of this equation by applying certain techniques. In this chapter we shall apply the following five techniques for solving a linear differential equation of second order.

(a) By changing the dependent variable when an integral included in the C.F. is known.

(b) By removing the first derivative and changing the dependent variable.

(c) By changing the independent variable.

(d) By the method of variation of parameters.

(e) By the method of undetermined coefficients.

6.2. TO SOLVE A LINEAR DIFFERENTIAL EQUATION OF SECOND ORDER BY CHANGING THE DEPENDENT VARIABLE WHEN AN INTEGRAL INCLUDED IN THE C.F. IS KNOWN

Consider the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad ...(1)$$

where P, Q and R are functions of x only.

Let $y = u$ be an integral included in the complementary function of the equation

$$\therefore \qquad \frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0$$

or

$$u_2 + Pu_1 + Qu = 0 \qquad ...(2)$$

Let

$$y = u.v$$

6.3. METHOD FOR FINDING PARTICULAR INTEGRAL OF $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$.

The given equation is $\quad \frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \qquad$ ...(1)

where P and Q are functions of x only

I. Consider that $y = e^{mx}$ is a solution of eqn. (1)

$$\therefore \quad \frac{dy}{dx} = me^{mx} \quad \text{and} \quad \frac{d^2y}{dx^2} = m^2e^{mx}$$

Then eqn. (1) becomes,

$$m^2 e^{mx} + P me^{mx} + Q e^{mx} = 0$$

or $\quad (m^2 + Pm + Q)\, e^{mx} = 0$

or $\quad m^2 + Pm + Q = 0$

or $\quad 1 + \frac{P}{m} + \frac{Q}{m^2} = 0 \qquad$ [*Dividing by* m^2]

Hence, we have

(i) $y = e^{mx}$ is a solution of eqn. (1) if $1 + \frac{P}{m} + \frac{Q}{m^2} = 0 \qquad$ [*K.U. 2017; M.D.U. 2017*]

(ii) $y = e^{ax}$ is a solution of eqn. (1) if $1 + \frac{P}{a} + \frac{Q}{a^2} = 0$

(iii) $y = e^{x}$ is a solution of eqn. (1) if $1 + P + Q = 0$

(iv) $y = e^{-x}$ is a solution of eqn. (1) if $1 - P + Q = 0$.

II. Consider that $y = x^m$ is a solution of eqn. (1)

$$\therefore \quad \frac{dy}{dx} = m x^{m-1} \quad \text{and} \quad \frac{d^2y}{dx^2} = m(m-1)x^{m-2}$$

Then eqn. (1) becomes

$$m(m-1)x^{m-2} + P.\, m x^{m-1} + Q x^m = 0$$

or $\quad m(m-1) + P.\, mx + Qx^2 = 0$

Hence, we have

(i) $y = x^m$ is a solution of eqn. (1) if $m(m-1) + mPx + Qx^2 = 0$

(ii) $y = x$ is a solution of eqn. (1) if $P + Qx = 0$

(iii) $y = x^2$ is a solution of eqn. (1) if $2 + 2Px + Qx^2 = 0$.

Working Rule :

(1) Find a solution of the equation $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0$ as discussed in article 6.3. Denote this solution by u. Sometimes it is given with the problem.

(2) Change the dependent variable y to v by putting $y=uv$. The reduced equation is of the form $\frac{d^2v}{dx^2}+P'\frac{dv}{dx}=Q'$.

(3) Put $\frac{dv}{dx}=p$ so that transformed equation is of the form $\frac{dp}{dx}+P'p=Q'$ where P′, Q′ are functions of x only.

(4) Solve the equation $\frac{dp}{dx}+P'p=Q'$ and find p *i.e.*, $\frac{dv}{dx}$. Integrate it and find v.

(5) Find the solution $y=uv$, which is the general solution of the given equation.

SOLVED EXAMPLES

Example 1.

Solve the differential equation

$$x^2\frac{d^2y}{dx^2}-(x^2+2x)\frac{dy}{dx}+(x+2)y=x^3e^x.$$ [K.U. 2012; M.D.U. 2003]

Solution. The given equation is

$$x^2\frac{d^2y}{dx^2}-(x^2+2x)\frac{dy}{dx}+(x+2)y=x^3e^x$$

or
$$\frac{d^2y}{dx^2}-\left(1+\frac{2}{x}\right)\frac{dy}{dx}+\left(\frac{1}{x}+\frac{2}{x^2}\right)y=xe^x \quad ...(1)$$

which is of the form $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$, where $P=-\left(1+\frac{2}{x}\right)$, $Q=\frac{1}{x}+\frac{2}{x^2}$ and $R=xe^x$

$$\therefore \quad P+Qx=-1-\frac{2}{x}+\frac{x}{x}+\frac{2}{x}=0$$

$\Rightarrow$ $y=x$ is a part of C.F.

Consider $y=vx$

$$\therefore \quad \frac{dy}{dx}=\frac{dv}{dx}x+v.$$

and
$$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2}x + 2\frac{dv}{dx}$$

Putting these values in (1), we get

$$\frac{d^2v}{dx^2}x + 2\frac{dv}{dx} - \left(1+\frac{2}{x}\right)\left(\frac{dv}{dx}x + v\right) + \left(\frac{1}{x}+\frac{2}{x^2}\right)v.x = xe^x$$

or
$$\frac{d^2v}{dx^2} - \frac{dv}{dx} = e^x$$

or
$$\frac{dp}{dx} - p = e^x \text{ where } p = \frac{dv}{dx} \quad ...(2)$$

which is a linear equation of type $\frac{dp}{dx} + P'p = Q'$

$$\text{I.F.} = e^{\int P'dx} = e^{\int -dx} = e^{-x}$$

Thus the solution of (2) is

$$pe^{-x} = \int e^x e^{-x}\,dx + c_1$$

or
$$pe^{-x} = x + c_1$$

or
$$p = xe^x + c_1e^x$$

or
$$\frac{dv}{dx} = xe^x + c_1e^x$$

ICAT3

Integrating, we have $\quad v = xe^x - e^x + c_1e^x + c_2$

Hence the complete solution is $y = vx$

i.e.,
$$y = x^2e^x - xe^x + c_1xe^x + c_2x.$$

Example 2.

Solve the differential equation

[M.D.U. 2019, 18]

$$(1-x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x(1-x^2)^{3/2}.$$

Solution. The given equation is

$$(1-x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x(1-x^2)^{3/2}.$$

or
$$\frac{d^2y}{dx^2} + \frac{x}{1-x^2}\frac{dy}{dx} - \frac{y}{1-x^2} = x(1-x^2)^{1/2} \quad ...(1)$$

which is of the form $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

where $P = \frac{x}{1-x^2}$, $Q = -\frac{1}{1-x^2}$ and $R = x(1-x^2)^{1/2}$

Here P, Q and R are functions of x only.

Also $\qquad P + Qx = 0$

$\therefore \qquad y = x$ is a part of C.F.

Consider $\qquad y = vx$

$$\therefore \qquad \frac{dy}{dx} = v(1) + x\frac{dv}{dx}$$

and
$$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2}x + 2\frac{dv}{dx}$$

Putting these values in (1), we get

$$\frac{d^2v}{dx^2}x + 2\frac{dv}{dx} + \frac{x}{1-x^2}\cdot\left[x\frac{dv}{dx} + v\right] - \frac{vx}{1-x^2} = x(1-x^2)^{1/2}$$

or
$$\frac{d^2v}{dx^2} + \left(\frac{x}{1-x^2} + \frac{2}{x}\right)\frac{dv}{dx} = \frac{x(1-x^2)^{1/2}}{x}$$

Putting $\frac{dv}{dx} = p$ and $\frac{d^2v}{dx^2} = \frac{dp}{dx}$, we get

$$\frac{dp}{dx} + \left(\frac{x}{1-x^2} + \frac{2}{x}\right)p = \sqrt{1-x^2} \qquad \text{...(2)}$$

which is a linear differential equation of the form $\frac{dp}{dx} + P'p = Q'$

where $P' = \frac{x}{1-x^2} + \frac{2}{x}$ and $Q' = \sqrt{1-x^2}$

$$\therefore \qquad \text{I.F.} = e^{\int P'dx} = e^{\int\left(\frac{x}{1-x^2}+\frac{2}{x}\right)dx}$$

$$= e^{-\frac{1}{2}\log(1-x^2) + 2\log x} = e^{\log(1-x^2)^{-1/2}.x^2} = (1-x^2)^{-\frac{1}{2}}.x^2$$

Thus the solution of (2) is

$$p(\text{I.F.}) = \int Q'(\text{I.F.})\,dx + c_1$$

$$p\,(1-x^2)^{-1/2}\,.\,x^2 = \int \sqrt{1-x^2}\,.\,\frac{x^2}{\sqrt{1-x^2}}\,dx + c_1$$

$$p\,\frac{x^2}{\sqrt{1-x^2}} = \frac{x^3}{3} + c_1$$

$$p = \frac{x}{3}\sqrt{1-x^2} + \frac{c_1\sqrt{1-x^2}}{x^2}$$

$$\frac{dv}{dx} = \frac{x}{3}\sqrt{1-x^2} + \frac{c_1\sqrt{1-x^2}}{x^2}$$

$\therefore$
$$\int dv = \int\left[\frac{x\sqrt{1-x^2}}{3} + \frac{c_1\sqrt{1-x^2}}{x^2}\right]dx + c_2$$

$$v = -\frac{1}{9}(1-x^2)^{3/2} + c_1\,(1-x^2)^{1/2}\left(-\frac{1}{x}\right) - c_1\int\frac{dx}{\sqrt{1-x^2}} + c_2$$

$$v = -\frac{1}{9}(1-x^2)^{3/2} - \frac{c_1}{x}\sqrt{1-x^2} - c_1\sin^{-1}x + c_2$$

$1 - x^2 = t.$
$-2x\,dx = dt$
$x\,dx = -\frac{dt}{2}.$
$-\frac{1}{2}\sqrt{t}\,dt,$
$-\frac{1}{2}\frac{t^{3/2}}{(3)/2} = \frac{1}{4}t^{3/2}.$

$c_1\left(\frac{\sqrt{1-x^2}}{x^2}\right)$
$c_1(1-x^2)^{1/2}.\frac{x^{-2+1}}{-2+1}$
$c_1(1-x^2)^{1/2}\left(\frac{-1}{x}\right)$

Hence the complete solution is $y = vx$

$$y = -\frac{1}{9}x(1-x^2)^{3/2} - c_1\sqrt{1-x^2} - c_1 x\sin^{-1}x + c_2 x.$$

Example 3.

Solve the differential equation

$$\frac{d^2y}{dx^2} - \cot x\,\frac{dy}{dx} - (1-\cot x)\,y = e^x \sin x.$$

[K.U. 2017, 16; M.D.U. 2017, 06]

Solution. The given equation is

$$\frac{d^2y}{dx^2} - \cot x\,\frac{dy}{dx} - (1-\cot x)\,y = e^x\sin x \qquad ...(1)$$

Comparing it with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, we have

$$P = -\cot x,\ Q = -1 + \cot x,\ R = e^x\sin x$$

Here $\quad P + Q + 1 = 0$

$\therefore$ $y = e^x$ is a part of C.F.

Consider $y = ve^x$

$$\therefore \quad \frac{dy}{dx} = e^x \frac{dv}{dx} + e^x v$$

and $$\frac{d^2y}{dx^2} = e^x \frac{d^2v}{dx^2} + 2\frac{dv}{dx}.e^x + ve^x$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$\frac{d^2v}{dx^2} + (2 - \cot x)\frac{dv}{dx} = \sin x$$

Putting $\frac{dv}{dx} = p$ and $\frac{d^2v}{dx^2} = \frac{dp}{dx}$, we get

$$\frac{dp}{dx} + (2 - \cot x)\, p = \sin x \quad ...(2)$$

which is a linear differential equation of the form

$$\frac{dp}{dx} + P'p = Q' \quad \text{where } P' = 2 - \cot x,\ Q' = \sin x$$

$$\therefore \quad \text{I.F.} = e^{\int P'dx} = e^{\int (2-\cot x)\,dx} = e^{2x - \log \sin x}$$

$$= \frac{e^{2x}}{e^{\log \sin x}} = \frac{e^{2x}}{\sin x}$$

Thus the solution of (2) is

$$p\,(\text{I.F.}) = \int Q'(\text{I.F.})\,dx + c_1$$

or $$p.\frac{e^{2x}}{\sin x} = \int \sin x.\frac{e^{2x}}{\sin x}\,dx + c_1$$

or $$p.\frac{e^{2x}}{\sin x} = \frac{e^{2x}}{2} + c_1$$

or $$p = \frac{e^{2x}}{2}.\frac{\sin x}{e^{2x}} + c_1\frac{\sin x}{e^{2x}}$$

or $$\frac{dv}{dx} = \frac{\sin x}{2} + c_1 e^{-2x}\sin x$$

$$\therefore \quad \int dv = \int\left[\frac{\sin x}{2} + c_1 e^{-2x}\sin x\right]dx$$

$$v = -\frac{\cos x}{2} + \frac{c_1 e^{-2x}}{5}[-2\sin x - \cos x] + c_2$$

Hence the complete solution is $y = v e^x$

i.e., $$y = -\frac{1}{2} e^x \cos x - \frac{c_1 e^{-x}}{5}[2\sin x + \cos x] + c_2 e^x.$$

Example 4.

Solve the differential equation

$$x(x\cos x - 2\sin x)\, y_2 + (x^2 + 2)\, y_1 \sin x - 2(x\sin x + \cos x)\, y = 0.$$

Solution. The given equation is [M.D.U. 2015]

$$x(x\cos x - 2\sin x)\frac{d^2y}{dx^2} + (x^2+2)\sin x \frac{dy}{dx} - 2(x\sin x + \cos x)\, y = 0$$

or $$\frac{d^2y}{dx^2} + \frac{(x^2+2)\sin x}{x[x\cos x - 2\sin x]}\frac{dy}{dx} - \frac{2[x\sin x + \cos x]}{x[x\cos x - 2\sin x]}\, y = 0 \qquad \text{...(1)}$$

Comparing (1) with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, we have

$$P = \frac{(x^2+2)\sin x}{x[x\cos x - 2\sin x]}, \quad Q = -\frac{2[x\sin x + \cos x]}{x[x\cos x - 2\sin x]}, \quad R = 0$$

Now, $$2 + 2Px + Qx^2 = 2 + \frac{2(x^2+2)\sin x}{x\cos x - 2\sin x} - \frac{2[x\sin x + \cos x]x}{x\cos x - 2\sin x}$$

$$= \frac{2[x\cos x - 2\sin x + x^2\sin x + 2\sin x - x^2\sin x - x\cos x]}{x\cos x - 2\sin x}$$

$$= 0$$

As $2 + 2Px + Qx^2 = 0$

$\therefore$ $y = x^2$ is a part of C.F.

Consider $y = vx^2$

$$\therefore \quad \frac{dy}{dx} = x^2\frac{dv}{dx} + 2vx$$

and $$\frac{d^2y}{dx^2} = x^2\frac{d^2v}{dx^2} + 4x\frac{dv}{dx} + 2v$$

Putting these values in (1), we have

$$x^2 \frac{d^2v}{dx^2} + 4x\frac{dv}{dx} + 2v + \frac{(x^2+2)\sin x}{x\,[x\cos x - 2\sin x]}\left[x^2\frac{dv}{dx} + 2vx\right] - \frac{2\,[x\sin x + \cos x]}{x\,[x\cos x - 2\sin x]}\,vx^2 = 0$$

or $$\frac{d^2v}{dx^2} + \frac{4}{x}\frac{dv}{dx} + \frac{2v}{x^2} + \frac{(x^2+2)\sin x}{x\,[x\cos x - 2\sin x]}\left[\frac{dv}{dx} + \frac{2v}{x}\right] - \frac{2(x\sin x + \cos x)}{x\,(x\cos x - 2\sin x)}\,v = 0$$

or $$\frac{d^2v}{dx^2} + \left[\frac{4}{x} + \frac{(x^2+2)\sin x}{x\,(x\cos x - 2\sin x)}\right]\frac{dv}{dx} + \frac{2v}{x}\left[\frac{1}{x} + \frac{(x^2+2)\sin x}{x\,(x\cos x - 2\sin x)} - \frac{x\sin x + \cos x}{x\cos x - 2\sin x}\right] = 0$$

or $$\frac{d^2v}{dx^2} + \left[\frac{4}{x} + \frac{(x^2+2)\sin x}{x\,(x\cos x - 2\sin x)}\right]\frac{dv}{dx} = 0$$

or $$\frac{dp}{dx} + \left[\frac{4}{x} + \frac{(x^2+2)\sin x}{x\,(x\cos x - 2\sin x)}\right]p = 0, \text{ where } p = \frac{dv}{dx}$$

or $$\frac{dp}{p} + \left[\frac{4}{x} + \frac{(x^2+2)\sin x}{x\,(x\cos x - 2\sin x)}\right]dx = 0$$ [*Variables separable*]

$\therefore$ $$\int \frac{dp}{p} + \int\left[\frac{4}{x} - \frac{-x^2\sin x - 2\sin x}{x^2\cos x - 2x\sin x}\right]dx = \log c_1$$

or $$\log p + 4\log x - \log(\overset{u\cdot v}{x^2\cos x} - \overset{uv}{2x\sin x}) = \log c_1$$

or $$\log\frac{px^4}{x^2\cos x - 2x\sin x} = \log c_1$$

or $$\frac{px^4}{x\,[x\cos x - 2\sin x]} = c_1$$

or $$p = \frac{c_1\,(x\cos x - 2\sin x)}{x^3}$$

i.e., $$\frac{dv}{dx} = \frac{c_1\,(x\cos x - 2\sin x)}{x^3}$$

$\therefore$ $$\int dv = c_1\int\left(\frac{\cos x}{x^2} - \frac{2}{x^3}\sin x\right)dx + c_2$$

i.e., $$v = c_1\,\frac{\sin x}{x^2} + c_2$$

Hence the complete solution is $y = vx^2$

$$y = \left(c_1 \frac{\sin x}{x^2} + c_2\right) x^2$$

i.e.,

$$y = c_1 \sin x + c_2 x^2.$$

or

Example 5.

Solve $\sin^2 x \frac{d^2 y}{dx^2} = 2y$, ***given that*** $y = \cot x$ ***is a solution.***

[M.D.U. 2013; 04 ; K.U. 2008]

Solution. The given equation can be written as

$$\frac{d^2y}{dx^2} = \frac{2y}{\sin^2 x} \qquad ...(1)$$

Let $y = v \cot x$ be the complete solution of (1)

$$\therefore \quad \frac{dy}{dx} = \frac{dv}{dx} \cot x - v \operatorname{cosec}^2 x$$

and
$$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2} \cot x - 2 \operatorname{cosec}^2 x \frac{dv}{dx} + 2v \operatorname{cosec}^2 x \cot x$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$\frac{d^2v}{dx^2} \cot x - 2 \operatorname{cosec}^2 x \frac{dv}{dx} + 2v \operatorname{cosec}^2 x \cot x = \frac{2v \cot x}{\sin^2 x}$$

or
$$\frac{d^2v}{dx^2} \cot x - 2 \operatorname{cosec}^2 x \frac{dv}{dx} = 0$$

or
$$\frac{dp}{dx} \cot x - (2 \operatorname{cosec}^2 x)p = 0, \text{ where } \frac{dv}{dx} = p$$

or
$$\frac{dp}{dx} - \frac{2 \operatorname{cosec}^2 x}{\cot x} p = 0$$

or
$$\frac{dp}{p} - 2 \frac{\operatorname{cosec}^2 x}{\cot x} dx = 0 \qquad \text{[Variables Separable]}$$

$$\therefore \quad \int \frac{dp}{p} + 2 \int \frac{-\operatorname{cosec}^2 x}{\cot x} dx = \log c_1$$

i.e.,
$$\log p + 2 \log \cot x = \log c_1$$

or
$$\log p (\cot x)^2 = \log c_1$$

or
$$p \cot^2 x = c_1 \Rightarrow p = c_1 \tan^2 x$$

$$\frac{dv}{dx} = c_1 \tan^2 x$$

i.e.,

$$dv = c_1 (\sec^2 x - 1)\, dx$$

or

Integrating,

$$v = c_1 (\tan x - x) + c_2$$

Hence the complete solution is $y = v \cot x$

i.e.,

$$y = c_1 (\tan x - x) \cot x + c_2 \cot x$$

or

$$y = c_1 (1 - x \cot x) + c_2 \cot x.$$

Example 6.

Solve the differential equation

$$(x+2)\frac{d^2y}{dx^2} - (2x+5)\frac{dy}{dx} + 2y = (x+1)\, e^x.$$ [K.U. 2015, 09; M.D.U. 2014]

Solution. The given equation is

$$\frac{d^2y}{dx^2} - \frac{2x+5}{x+2}\frac{dy}{dx} + \frac{2y}{x+2} = \frac{(x+1)\, e^x}{x+2} \quad ...(1)$$

Comparing (1) with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, we have

$$P = -\left(\frac{2x+5}{x+2}\right), \quad Q = \frac{2}{x+2}, \quad R = \frac{(x+1)\, e^x}{x+2}$$

Here

$$\frac{P}{2} + \frac{Q}{2^2} + 1 = \frac{-2x-5}{2(x+2)} + \frac{1}{2(x+2)} + 1$$

$$= \frac{-2x-5+1+2x+4}{2(x+2)} = 0$$

$\therefore \quad y = e^{2x}$ is a part of C.F.

Consider

$$y = ve^{2x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dv}{dx} e^{2x} + 2ve^{2x}$$

and

$$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2} e^{2x} + 4\frac{dv}{dx} e^{2x} + 4ve^{2x}$$

Putting these values in (1), we get

$$\frac{d^2v}{dx^2} e^{2x} + 4\frac{dv}{dx} e^{2x} + 4ve^{2x} - \frac{2x+5}{x+2}\left[\frac{dv}{dx} e^{2x} + 2ve^{2x}\right] + \frac{2}{x+2}(ve^{2x}) = \frac{(x+1)\, e^x}{x+2}$$

Multiplying by e^{-2x}, we get

$$\frac{d^2v}{dx^2} + 4\frac{dv}{dx} + 4v - \frac{2x+5}{x+2}\left(\frac{dv}{dx} + 2v\right) + \frac{2v}{x+2} = \frac{(x+1)e^{-x}}{x+2}$$

or

$$\frac{d^2v}{dx^2} + \frac{2x+3}{x+2}\frac{dv}{dx} = \frac{x+1}{x+2}e^{-x}$$

or

$$\frac{dp}{dx} + \frac{2x+3}{x+2}p = \frac{x+1}{x+2}e^{-x} \qquad ...(2)$$

where $p = \frac{dv}{dx}$

Equation (2) is of the form $\frac{dp}{dx} + P'p = Q'$ where $P' = \frac{2x+3}{x+2}$, $Q' = \frac{x+1}{x+2}e^{-x}$

$\frac{2x}{x+2} = \frac{2x^2}{2} = \frac{x^2}{2}$

$\therefore$

$$\text{I.F.} = e^{\int P' dx} = e^{\int \frac{2x+3}{x+2} dx}$$

$$= e^{\int\left(2 - \frac{1}{x+2}\right)dx} = e^{2x - \log(x+2)}$$

$$= e^{2x} . e^{-\log(x+2)} = e^{2x} . e^{\log(x+2)^{-1}} = \frac{e^{2x}}{x+2}$$

log

Thus the solution of (2) is

$$p.(\text{I.F.}) = \int Q'(\text{I.F.})\, dx + c_1$$

or

$$p\frac{e^{2x}}{x+2} = \int \frac{x+1}{x+2}e^{-x} . \frac{e^{2x}}{x+2} dx + c_1$$

$$= \int \frac{x+1}{(x+2)^2} e^x dx + c_1$$

using partial fraction

$$= \int \left[\frac{1}{x+2} - \frac{1}{(x+2)^2}\right] e^x dx + c_1 = \frac{e^x}{x+2} + c_1$$

$\therefore$

$$p = e^{-x} + c_1 e^{-2x}(x+2)$$

i.e.,

$$\frac{dv}{dx} = e^{-x} + c_1 e^{-2x}(x+2)$$

Integrating,

$$v = -e^{-x} + c_1\left[\frac{e^{-2x}(x+2)}{-2} - \int \frac{e^{-2x}}{-2} dx\right] + c_2$$

$$= -e^{-x} + c_1\left[\frac{e^{-2x}(x+2)}{-2} - \frac{e^{-2x}}{4}\right] + c_2$$

$$= -e^{-x} - \frac{c_1}{4}(2x+5)\,e^{-2x} + c_2$$

Hence the complete solution is $y = v\,e^{2x}$

i.e., $$y = -e^{x} - \frac{c_1}{4}(2x+5) + c_2\,e^{2x}.$$

EXERCISE 6.1

Solve the following differential equations :

1. $x\dfrac{d^2y}{dx^2} - (2x-1)\dfrac{dy}{dx} + (x-1)\,y = 0$

2. $x^2\dfrac{d^2y}{dx^2} - 2x(1+x)\dfrac{dy}{dx} + 2(1+x)\,y = x^3$ [K.U. 2015; M.D.U. 2014, 13, 12, 09, 0

3. $x\dfrac{dy}{dx} - y = (x-1)\left[\dfrac{d^2y}{dx^2} - x + 1\right]$

4. $x\dfrac{d^2y}{dx^2} - 2(x+1)\dfrac{dy}{dx} + (x+2)\,y = (x-2)\,e^{x}$ [M.D.U. 2007

5. $(x+2)\dfrac{d^2y}{dx^2} - (2x+5)\dfrac{dy}{dx} + 2y = (x+1)\,e^{x}$ [K.U. 2016]

6. $x\dfrac{d^2y}{dx^2} - 2(x+1)\dfrac{dy}{dx} + (x+2)y = (x-2)e^{2x}$

7. $(x+1)\dfrac{d^2y}{dx^2} - 2(x+3)\dfrac{dy}{dx} + (x+5)\,y = e^{x}$ [K.U. 2013]

8. $(x\sin x + \cos x)\dfrac{d^2y}{dx^2} - x\cos x\dfrac{dy}{dx} + y\cos x = \sin x\,(x\sin x + \cos x)^2$

9. $(x\sin x + \cos x)\dfrac{d^2y}{dx^2} - x\cos x\dfrac{dy}{dx} + y\cos x = 0$

10. $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + (1-x)\,y = x^2e^{-x}$

11. $x\dfrac{d^2y}{dx^2} - (x+2)\dfrac{dy}{dx} + 2y = x^3$ [M.D.U. 2015

12. $x\dfrac{d^2y}{dx^2} + (x-2)\dfrac{dy}{dx} - 2y = x^3$

13. $\dfrac{d^2y}{dx^2} - x^2\dfrac{dy}{dx} + xy = x$

14. $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = 8x^3$

15. $x\dfrac{d^2y}{dx^2} + (1-x)\dfrac{dy}{dx} - y = e^{x}$ [M.D.U. 2016

16. $x\dfrac{d^2y}{dx^2} + (x-1)\dfrac{dy}{dx} - y = x^2$

17. $\dfrac{d^2y}{dx^2} + \left(1 + \dfrac{2}{x}\cot x - \dfrac{2}{x^2}\right)y = x\cos x$, given that $\dfrac{\sin x}{x}$ is a C.F.

18. $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - 9y = 0$ given that $y = x^3$ is a part of solution. [M.D.U. 2011]

19. $x\frac{d^2y}{dx^2} - (2x+1)\frac{dy}{dx} + (x+1)y = (x^2+x-1)e^{2x}$

20. $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$ given that $y = \left(x + \frac{1}{x}\right)$ is one integral. [K.U. 2011]

21. $\frac{d^2y}{dx^2} + (1-\cot x)\frac{dy}{dx} - y\cot x = \sin^2 x$

ANSWERS

1. $y = e^x[c_1 \log x + c_2]$

2. $y = -\frac{1}{2}x^2 + \frac{c_1}{2}xe^{2x} + c_2x$

3. $y = c_1e^x + c_2x - (1+x^2)$

4. $y = -\frac{1}{2}x^2e^x + xe^x + \frac{1}{3}c_1x^3e^x + c_2e^x$

5. $y = -e^x - \frac{1}{4}c_1(2x+5) + c_2e^{2x}$

6. $y = \frac{1}{3}c_1x^3e^x + c_2e^x + e^{2x}$

7. $y = \frac{1}{5}c_1e^x(x+1)^5 - \frac{1}{4}xe^x + c_2e^x$

8. $y = \frac{x}{2}\cos^2 x - \frac{1}{2}\sin 2x - c_1\cos x + c_2 x$

9. $y = -c_1\cos x + c_2 x$

10. $y = c_2e^x + c_1(2x+1)e^{-x} - \frac{e^{-x}}{4}[2x^2+2x+1]$

11. $y = -x^3 - (c_1+3)(x^2+2x+2) + c_2e^x$

12. $y = x^3 + (c_1-3)(x^2-2x+2) + c_2e^{-x}$

13. $y = 1 + c_1x\int x^{-2}e^{x^3/3}\,dx + c_2x$

14. $y = c_1x + \frac{c_2}{x} + x^3$

15. $y = e^x\log x + c_1e^x\int \frac{e^{-x}}{x}\,dx + c_2e^x$

16. $y = c_1(x-1) + c_2e^{-x} + x^2 - 2x + 2.$

17. $\left(c_1 - \frac{1}{4}\right)\left[-x\cos x + 2\sin x\log\sin x - \frac{2\sin x}{x}\int \log\sin x\,dx\right] + \frac{x^2}{6}\sin x + \frac{c_2\sin x}{x}$

18. $y = c_1x^3 + c_2x^{-3}$

19. $y = xe^{2x} + c_1x^2e^x + c_2e^x$

20. $y = \frac{c_1}{x} + c_2\left(x + \frac{1}{x}\right)$

21. $y = c_2e^{-x} + c_1(\sin x - \cos x) - \frac{1}{10}[\sin 2x - 2\cos 2x]$

6.4. TO SOLVE A LINEAR DIFFERENTIAL EQUATION OF SECOND ORDER BY REMOVING THE FIRST DERIVATIVE AND CHANGING THE DEPENDENT VARIABLE

Consider the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad ...(1)$$

where P, Q and R are functions of x only.

This method is used when a particular integral of $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ is neither given nor can be easily found.

Let $y = uv$, where u, v are functions of x

$$\therefore \quad \frac{dy}{dx} = u\frac{dv}{dx} + \frac{du}{dx}.v$$

and
$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2}$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + v\frac{d^2u}{dx^2} + P\left[u\frac{dv}{dx} + v\frac{du}{dx}\right] + Q.uv = R$$

or
$$u.\frac{d^2v}{dx^2} + \left(Pu + 2\frac{du}{dx}\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = R$$

or
$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} + \left(\frac{1}{u}\frac{d^2u}{dx^2} + \frac{P}{u}\frac{du}{dx} + Q\right)v = \frac{R}{u} \qquad ...(2)$$

Choose u such that coefficient of $\frac{dv}{dx}$ is zero.

$$\therefore \quad P + \frac{2}{u}\frac{du}{dx} = 0$$

or
$$\frac{2}{u}\frac{du}{dx} = -P \quad i.e., \quad \frac{du}{u} = -\frac{P}{2}dx$$

, $\frac{du}{dx} = \frac{-Pu}{2}$

Integrating, $\log u = -\frac{1}{2}\int P\,dx \Rightarrow u = e^{-\frac{1}{2}\int P\,dx}$

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$\therefore$ Equation (2) becomes

$$\frac{d^2v}{dx^2}+\left[\frac{1}{u}\left(-\frac{u}{2}\frac{dP}{dx}-\frac{P}{2}\frac{du}{dx}\right)+\frac{P}{u}\frac{du}{dx}+Q\right]v = Re^{\frac{1}{2}\int P\,dx}$$

or

$$\frac{d^2v}{dx^2}+\left[-\frac{1}{2}\frac{dP}{dx}-\frac{P}{2u}\left(-\frac{P}{2}u\right)+\frac{P}{u}\left(-\frac{Pu}{2}\right)+Q\right]v = R.e^{\frac{1}{2}\int P\,dx}$$

or

$$\frac{d^2v}{dx^2}+\left[Q-\frac{1}{2}\frac{dP}{dx}-\frac{1}{4}P^2\right]v = R.e^{\frac{1}{2}\int P\,dx}$$

or

$$\frac{d^2v}{dx^2}+P'v = Q' \qquad ...(3)$$

where $P' = Q-\frac{1}{2}\frac{dP}{dx}-\frac{1}{4}P^2$ and $Q' = R\,e^{\frac{1}{2}\int P\,dx} = \frac{R}{u}$

Equation (3) is the normal form of equation (1) and can be easily integrated to find the value of v.

Hence the solution $y = uv$ is known. This is the general solution of the given equation and contains two arbitrary constants.

Note.

Transformation of equation (1) to the form (3) is known as removal of the first derivative.

Working Rule :

(1) Find $P' = Q-\frac{1}{4}P^2-\frac{1}{2}\frac{dP}{dx}$. We apply this method when P' is either constant or of the form $\frac{\text{constant}}{x^2}$.

(2) Find $e^{-\frac{1}{2}\int P\,dx}$ and denote it by u.

(3) After putting $y = uv$, the given equation transforms to the form $\frac{d^2v}{dx^2}+P'v = Q'$ where $Q' = \frac{R}{u}$.

(4) Solve the equation $\frac{d^2v}{dx^2}+P'v = Q'$ and find v.

(5) The general solution is $y = uv$ and it contains two arbitrary constants.

SOLVED EXAMPLES

Example 1.

Solve $\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 5y = 0$ *by removing the first derivative.*

[K.U. 2016, 15, 12; M.D.U. 2009, 08, 03]

Solution. The given equation is

$$\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 5y = 0 \qquad ...(1)$$

Comparing (1) with $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, we get

$$P = -2\tan x, \quad Q = 5, \quad R = 0$$

Putting $y = uv$ in the given equation, we get

$$\frac{d^2v}{dx^2} + P'v = Q' \qquad ...(2)$$

where
$$u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int -2\tan x\,dx}$$

$$= e^{\int \tan x\,dx} = e^{\log\sec x} = \sec x$$

$$P' = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$$

$$= 5 - \frac{1}{2}\frac{d}{dx}(-2\tan x) - \frac{1}{4}(4\tan^2 x)$$

$$= 5 + \sec^2 x - \tan^2 x = 5 + (1 + \tan^2 x) - \tan^2 x = 6$$

and
$$Q' = \frac{R}{u} = 0$$

$\therefore$ Equation (2) becomes $\frac{d^2v}{dx^2} + 6v = 0$

or
$$(D^2 + 6)v = 0$$

$\therefore$ Auxiliary equation is

$$D^2 + 6 = 0 \Rightarrow D^2 = -6 \Rightarrow D = \pm\sqrt{6}\,i$$

Hence
$$v = c_1\cos\sqrt{6}x + c_2\sin\sqrt{6}x$$

The solution of the given equation is $y = uv$

or
$$y = \sec x\,[c_1\cos\sqrt{6}x + c_2\sin\sqrt{6}x].$$

Example 2.

Solve $\left[\frac{d^2y}{dx^2}+y\right]\cot x+2\left[\frac{dy}{dx}+y\tan x\right]=\sec x.$ **[K.U. 2019, 07]**

Solution. The given equation is

$$\left(\frac{d^2y}{dx^2}+y\right)\cot x+2\left(\frac{dy}{dx}+y\tan x\right) = \sec x$$

or

$$\frac{d^2y}{dx^2}+y+2\tan x\left(\frac{dy}{dx}+y\tan x\right) = \tan x\sec x$$

or

$$\frac{d^2y}{dx^2}+2\tan x\frac{dy}{dx}+(2\tan^2 x+1)y = \tan x\sec x \qquad ...(1)$$

Comparing (1) with $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$, we get

$$P=2\tan x,\quad Q=1+2\tan^2 x \quad \text{and} \quad R=\tan x\sec x.$$

Putting $y=uv$ in (1), we get

$$\frac{d^2v}{dx^2}+P'v=Q' \qquad ...(2)$$

where

$$u = e^{-\frac{1}{2}\int P\,dx}=e^{-\frac{1}{2}\int 2\tan x\,dx}$$

$$= e^{-\log\sec x} = (\sec x)^{-1} = \cos x$$

$$P' = Q-\frac{1}{2}\frac{dP}{dx}-\frac{P^2}{4}$$

$$=1+2\tan^2 x-\frac{1}{2}\frac{d}{dx}(2\tan x)-\frac{1}{4}(4\tan^2 x)$$

$$=1+2\tan^2 x-\sec^2 x-\tan^2 x$$

$$=1+\tan^2 x-\sec^2 x=\sec^2 x-\sec^2 x=0$$

and

$$Q' = \frac{R}{u}=\frac{\tan x\sec x}{\cos x}=\tan x\sec^2 x.$$

$\therefore$ Equation (2) becomes

$$\frac{d^2v}{dx^2}+0 = \tan x\sec^2 x \qquad ...(3)$$

or

$$D^2v = \tan x\sec^2 x$$

Auxiliary equation is $\quad D^2=0 \Rightarrow D=0,0$

Now, $\text{C.F.} = (c_1 + c_2 x)\, e^{0x} = c_1 + c_2 x$

and $$\text{P.I.} = \frac{1}{D^2} \tan x \sec^2 x = \frac{1}{D} \frac{\sec^2 x}{2} = \frac{\tan x}{2}$$

$\therefore$ $$v = (c_1 + c_2 x) + \frac{\tan x}{2}$$

Hence, the solution of the given equation is $y = uv$

i.e., $$y = \cos x \left[c_1 + c_2 x + \frac{\tan x}{2} \right] = (c_1 + c_2 x) \cos x + \frac{\sin x}{2}.$$

Example 3.

Solve $\dfrac{d}{dx}\left(\cos^2 x \dfrac{dy}{dx}\right) + y \cos^2 x = 0.$ *[M.D.U. 2015, 08; K.U. 2013, 06, 04]*

Solution. The given equation is $\dfrac{d}{dx}\left(\cos^2 x \dfrac{dy}{dx}\right) + y \cos^2 x = 0$ $\cos^2 x = -2\sin x \cos x$

or $$\cos^2 x \frac{d^2 y}{dx^2} + (-2 \sin x \cos x) \frac{dy}{dx} + y \cos^2 x = 0$$

or $$\frac{d^2 y}{dx^2} - 2 \tan x \frac{dy}{dx} + y = 0 \qquad ...(1)$$

Comparing (1) with $\dfrac{d^2 y}{dx^2} + P \dfrac{dy}{dx} + Qy = R$, we get

$$P = -2 \tan x, \quad Q = 1 \quad \text{and} \quad R = 0$$

Putting $y = uv$ in (1), we get $\dfrac{d^2 v}{dx^2} + P' v = Q'$ ...(2)

where $$u = e^{-\frac{1}{2}\int P\, dx} = e^{-\frac{1}{2}\int -2 \tan x\, dx} = e^{\log \sec x} = \sec x$$

$$P' = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = 1 - \frac{1}{2}\frac{d}{dx}(-2 \tan x) - \frac{1}{4}(4 \tan^2 x)$$

$$= 1 + \sec^2 x - \tan^2 x = 1 + (1 + \tan^2 x) - \tan^2 x = 2$$

and $$Q' = \frac{R}{u} = 0.$$

Hence equation (2) becomes

$$\frac{d^2 v}{dx^2} + 2v = 0 \qquad ...(3)$$

or $$(D^2 + 2)\, v = 0$$

$\therefore$ Auxiliary equation is $D^2 + 2 = 0$

$$D^2 = -2 \quad \Rightarrow \quad D = \pm \sqrt{2}\, i$$

$\therefore$ $$v = c_1 \cos \sqrt{2}\, x + c_2 \sin \sqrt{2}\, x$$

$\therefore$ Hence, the solution of the given equation is $y = uv$

i.e., $$y = \sec x\, [c_1 \cos \sqrt{2}\, x + c_2 \sin \sqrt{2}\, x].$$

EXERCISE 6.2

Solve the following equations by removing the first derivative :

1. $\dfrac{d^2y}{dx^2} - \dfrac{2}{x}\left(\dfrac{dy}{dx}\right) + \left(a^2 + \dfrac{2}{x^2}\right) y = 0$ [M.D.U. 2005]

2. $(x^3 - 2x^2)\dfrac{d^2y}{dx^2} + 2x^2 \dfrac{dy}{dx} - 12(x-2)\, y = 0$

3. $\dfrac{d^2y}{dx^2} + \dfrac{1}{x^{1/3}}\dfrac{dy}{dx} + \left(\dfrac{1}{4x^{2/3}} - \dfrac{1}{6x^{4/3}} - \dfrac{6}{x^2}\right) y = 0$

4. $x^2 \dfrac{d^2y}{dx^2} - 2(x^2 + x)\dfrac{dy}{dx} + (x^2 + 2x + 2)\, y = 0$

5. $\dfrac{d^2y}{dx^2} - 2bx \dfrac{dy}{dx} + b^2x^2y = 0$

6. $\dfrac{d^2y}{dx^2} + \dfrac{2}{x}\dfrac{dy}{dx} + n^2y = 0$ [M.D.U. 2017, 15,14; K.U. 2017]

7. $\dfrac{d^2y}{dx^2} + 4x \dfrac{dy}{dx} + 4x^2y = 0$

8. $x \dfrac{d}{dx}\left[x \dfrac{dy}{dx} - y\right] - 2x \dfrac{dy}{dx} + 2y + x^2y = 0$ [M.D.U. 2017]

9. $\dfrac{d^2y}{dx^2} - \dfrac{2}{x}\dfrac{dy}{dx} + \left(n^2 + \dfrac{2}{x^2}\right) y = 0$ [K.U. 2014; M.D.U. 2013]

10. $\dfrac{d^2y}{dx^2} - 4x \dfrac{dy}{dx} + (4x^2 - 3)\, y = e^{x^2}$ [K.U. 2015]

11. (i) $\dfrac{d^2y}{dx^2} - 2\tan x \dfrac{dy}{dx} + 5y = (\sec x)\, e^x$ (ii) $\dfrac{d^2y}{dx^2} - 2\tan x \dfrac{dy}{dx} + 5y = (\sec x) \sin 2x$ [M.D.U. 2016]

12. $\dfrac{d^2y}{dx^2} - 2\tan x \dfrac{dy}{dx} - (a^2 + 1)y = \sec x . e^x$ [K.U. 2016; M.D.U. 2014]

13. $\dfrac{d^2y}{dx^2} + \dfrac{2}{x}\dfrac{dy}{dx} = n^2 y$

14. $\dfrac{d^2y}{dx^2} - 4x \dfrac{dy}{dx} + (4x^2 - 1)\, y = e^{x^2}[5 - 3\cos 2x]$

15. $\dfrac{d^2y}{dx^2} - 4x \dfrac{dy}{dx} + (4x^2 - 1)\, y = -3e^{x^2} \sin 2x.$

16. $\dfrac{d^2y}{dx^2} - 4x \dfrac{dy}{dx} + (4x^2 - 1)\, y = -3e^{x^2}[\sin 2x + 5e^{-2x} + 6]$ [M.D.U. 2012]

17. $\dfrac{d^2y}{dx^2}-\dfrac{2}{x}\dfrac{dy}{dx}+\left(1+\dfrac{2}{x^2}\right)y=xe^x$

18. $\dfrac{d^2y}{dx^2}-2bx\dfrac{dy}{dx}+b^2x^2y=x$

19. $x^2(\log x)^2\dfrac{d^2y}{dx^2}-2x\log x\dfrac{dy}{dx}+[2+\log x-2(\log x)^2]\,y=(\log x)^3\,.\,x^2$

20. $\dfrac{d^2y}{dx^2}-\dfrac{1}{\sqrt{x}}\dfrac{dy}{dx}+\dfrac{1}{4x^2}(-8+\sqrt{x}+x)y=0$

21. $\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}+(x^2+1)\,y=x^3+3x$ [M.D.U. 2005]

22. $\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}+(x^2+5)\,y=xe^{-\frac{1}{2}x^2}$

ANSWERS

1. $y=x\,[c_1\cos ax+c_2\sin ax]$

2. $y=\dfrac{c_1x^4+c_2x^{-3}}{(x-2)}$

3. $y=e^{-\frac{3}{4}x^{2/3}}\,[c_1x^3+c_2x^{-2}]$

4. $y=x\,e^x\,[c_1x+c_2]$

5. $y=e^{\frac{1}{2}bx^2}\,[c_1\cos\sqrt{b}\,x+c_2\sin\sqrt{b}\,x]$

6. $y=\dfrac{1}{x}[c_1\cos nx+c_2\sin nx]$

7. $y=e^{-x^2}\left[c_1e^{\sqrt{2}\,x}+c_2e^{-\sqrt{2}\,x}\right]$

8. $y=x\,[c_1\cos x+c_2\sin x]$

9. $y=x\,[c_1\cos nx+c_2\sin nx]$

10. $y=e^{x^2}\,[c_1e^x+c_2e^{-x}-1]$

11. (i) $y=\sec x\left[c_1\cos\sqrt{6}\,x+c_2\sin\sqrt{6}\,x+\dfrac{1}{7}e^x\right]$ (ii) $y=\sec x\left[c_1\cos\sqrt{6}\,x+c_2\sin\sqrt{6}\,x+\dfrac{\sin 2x}{2}\right]$

12. $y=(c_1e^{ax}+c_2e^{-ax})\sec x+\dfrac{e^x\sec x}{1-a^2}$

13. $y=\dfrac{1}{x}[c_1e^{nx}+c_2e^{-nx}]$

14. $y=e^{x^2}\,[c_1\cos x+c_2\sin x+\cos 2x+5]$

15. $y=e^{x^2}\,[c_1\cos x+c_2\sin x+\sin 2x]$

16. $y=e^{x^2}\,[c_1\cos x+c_2\sin x+\sin 2x-3e^{-2x}-18]$

17. $y=x\left[c_1\cos x+c_2\sin x+\dfrac{1}{2}e^x\right]$

18. $y=e^{\frac{1}{2}bx^2}\,[c_1\cos\sqrt{b}\,x+c_2\sin\sqrt{b}\,x]+e^{\frac{1}{2}bx^2}\left[\dfrac{1}{D^2+b}\left(xe^{-\frac{1}{2}bx^2}\right)\right]$

19. $y=(\log x)\,[c_1x^2+c_2x^{-1}]+\dfrac{1}{3}[x\log x]^2$

20. $y=e^{\sqrt{x}}\,[c_1x^2+c_2x^{-1}]$

21. $y=x+(c_1x+c_2)e^{-\frac{x^2}{2}}$

22. $y=e^{-\frac{x^2}{2}}\left[(c_1\cos 2x+c_2\sin 2x)+\dfrac{x}{4}\right]$

Working Rule :

(1) Find two independent solutions of the equation $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ and denote them by y_1 and y_2.

(2) Put C.F. $= Ay_1 + By_2$, where A and B are arbitrary constants.

(3) Find $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$, $u = -\int \frac{y_2 R}{W} dx + c_1$, $v = \int \frac{y_1 R}{W} dx + c_2$

(4) Replace arbitrary constants A and B in C.F. $= Ay_1 + By_2$ by functions u and v so that the complete solution is $y = uy_1 + vy_2$.

SOLVED EXAMPLES

Example 1.

Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2} + 4y = \tan 2x$.

[K.U. 2014, 04; M.D.U. 2009, 07, 04]

Solution. The given equation is

$$\frac{d^2y}{dx^2} + 4y = \tan 2x \qquad ...(1)$$

or $$(D^2 + 4)\,y = \tan 2x \quad \text{where} \quad D = \frac{d}{dx}$$

$\therefore$ Auxiliary equation is $D^2 + 4 = 0$

or $$D^2 = -4 \Rightarrow D = \pm 2i$$

$\therefore$ $$\text{C.F.} = A\cos 2x + B\sin 2x$$

Let the complete solution of (1) be

$$y = u\cos 2x + v\sin 2x \qquad ...(2)$$

where u, v are unknown functions of x

or $$y = uy_1 + vy_2, \quad \text{where } y_1 = \cos 2x \text{ and } y_2 = \sin 2x$$

Let $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

i.e., $$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x$$

$$= 2(\cos^2 2x + \sin^2 2x) = 2.$$

Now, $$u = -\int \frac{y_2 R}{W} dx + c_1 \quad \text{where } R = \tan 2x$$

$$= -\int \frac{(\sin 2x)\tan 2x}{2}\,dx + c_1 = -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x}\,dx + c_1$$

$$= -\frac{1}{2}\int \left(\frac{1-\cos^2 2x}{\cos 2x}\right) dx + c_1$$

$$= -\frac{1}{2}\int (\sec 2x - \cos 2x)\,dx + c_1$$

$$= -\frac{1}{2}\frac{[\log(\sec 2x + \tan 2x) - \sin 2x]}{2} + c_1$$

and $$v = \int \frac{y_1 R}{W}\,dx + c_2 = \int \frac{\cos 2x \,.\, \tan 2x}{2}\,dx + c_2$$

$$= \frac{1}{2}\int \sin 2x\,dx + c_2 = -\frac{\cos 2x}{4} + c_2$$

From (2), $y = u\cos 2x + v\sin 2x$

$$= -\frac{1}{4}[\log(\sec 2x + \tan 2x) - \sin 2x]\cos 2x$$

$$- \frac{\cos 2x}{4}\sin 2x + c_1\cos 2x + c_2\sin 2x$$

or $$y = -\frac{\cos 2x}{4}[\log(\sec 2x + \tan 2x) - \sin 2x + \sin 2x] + c_1\cos 2x + c_2\sin 2x.$$

Hence the complete solution of (1) is

$$y = c_1\cos 2x + c_2\sin 2x - \frac{\cos 2x}{4}[\log(\sec 2x + \tan 2x)].$$

Example 2. ***Apply the method of variation of parameters to solve*** $\frac{d^2y}{dx^2} + n^2 y = \sec nx.$

[M.D.U. 2017, 16, 14; K.U. 2017, 15, 13]

Solution. The given equation is $\frac{d^2y}{dx^2} + n^2 y = \sec nx$ ...(1)

or $$(D^2 + n^2)\,y = \sec nx$$

Auxiliary equation is $D^2 + n^2 = 0$

or $D^2 = -n^2 \Rightarrow D = \pm in$

$\therefore$ C.F. $= A\cos nx + B\sin nx$

Replacing A, B by unknown functions $u(x)$ and $v(x)$, the complete primitive of (1) is

$$y = u\cos nx + v\sin nx \quad ...(2)$$

or $y = uy_1 + vy_2$, where $y_1 = \cos nx$ and $y_2 = \sin nx$

Let $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos nx & \sin nx \\ -n\sin nx & n\cos nx \end{vmatrix}$$

$$= n(\cos^2 nx + \sin^2 nx) = n(1) = n$$

Now, $\quad u = -\int \dfrac{y_2 R}{W}\,dx + c_1$, where $R = \sec nx$

$$= -\int \frac{(\sin nx)\sec nx}{n}\,dx + c_1$$

$$= -\frac{1}{n}\int \tan nx\,dx + c_1 = -\frac{1}{n}\left(-\frac{\log \cos nx}{n}\right) + c_1$$

i.e., $\quad u = \dfrac{1}{n^2}\log \cos nx + c_1$

Also, $\quad v = \int \dfrac{y_1 R}{W}\,dx + c_2 = \int \dfrac{\cos nx \sec nx}{n}\,dx + c_2$

$$= \int \frac{1}{n}\,dx + c_2 = \frac{x}{n} + c_2$$

From (2), $\quad y = u \cos nx + v \sin nx$

or $\quad y = \left[\dfrac{1}{n^2}\log \cos nx + c_1\right]\cos nx + \left(\dfrac{x}{n} + c_2\right)\sin nx$

or $\quad y = \dfrac{1}{n^2}(\log \cos nx)\cos nx + \dfrac{x}{n}\sin nx + c_1 \cos nx + c_2 \sin nx$.

which is the complete solution of the given equation (1).

Example 3.

Apply the method of variation of parameters to solve the equation

$$(1-x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = (1-x)^2.$$

[K.U. 2018, 16, 09, 07; M.D.U. 2015]

Solution. The given equation is

$$\frac{d^2y}{dx^2} + \frac{x}{1-x}\frac{dy}{dx} - \frac{y}{1-x} = 1-x \quad \ldots(1)$$

→ if this condition is not finding Sol^n this method C.F.

Comparing (1) with $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = R$, we get

$$P = \frac{x}{1-x}, \quad Q = -\frac{1}{1-x} \quad \text{and} \quad R = 1-x$$

As $\quad P + Qx = \dfrac{x}{1-x} - \dfrac{x}{1-x} = 0$

$\therefore \quad y = x$ is a part of C.F.

Also $\quad 1 + P + Q = 1 + \dfrac{x}{1-x} - \dfrac{1}{1-x} = \dfrac{1 - x + x - 1}{1-x} = 0$

$\therefore \quad y = e^x$ is a part of C.F.

Thus C.F. $= Ax + Be^x$, where A, B are arbitrary constants.

Replacing A, B by unknown functions u, v, the complete primitive of (1) is

$$y = ux + ve^x \quad \text{...(2)}$$

or
$$y = u\phi_1(x) + v\phi_2(x), \quad \text{where} \quad \phi_1(x) = x \quad \text{and} \quad \phi_2(x) = e^x$$

$$W = \begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{vmatrix} = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = xe^x - e^x = (x-1)\,e^x.$$

Now, $\quad u = -\displaystyle\int \frac{\phi_2(x)\,R}{W}\,dx + c_1$, where $R = (1-x)$

$$= -\int \frac{e^x\,(1-x)}{e^x\,(x-1)}\,dx + c_1 = \int 1.dx + c_1 = x + c_1$$

$\int u \cdot v = u \int v - \int \left[\frac{d}{dx}(u) \int v\right]$

$$v = \int \frac{\phi_1(x)\,R}{W}\,dx + c_2 = \int \frac{x\,(1-x)}{e^x\,(x-1)}\,dx + c_2$$

$$= -\int xe^{-x}\,dx + c_2 = -[-xe^{-x} - e^{-x}] + c_2 = e^{-x}(1+x) + c_2.$$

Putting the values of u and v in (2), we get

$$y = (x + c_1)\,x + [e^{-x}(1+x) + c_2]\,e^x$$

or
$$y = c_1 x + c_2 e^x + x^2 + (1+x)$$

which is the complete solution of the given equation.

Example 4.

Solve $x^2 \dfrac{d^2y}{dx^2} - 2x(1+x)\dfrac{dy}{dx} + 2(1+x)\,y = x^3$ ***by variation of parameters method.***

Solution. The given equation is

$$x^2 \frac{d^2y}{dx^2} - 2x\,(1+x)\frac{dy}{dx} + 2(1+x)\,y = x^3$$

or
$$\frac{d^2y}{dx^2} - \frac{2}{x}(1+x)\frac{dy}{dx} + \frac{2}{x^2}(1+x)\,y = x \quad \text{...(1)}$$

Comparing (1) with $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = R$, we get

$$P = -\frac{2}{x}(1+x), \quad Q = \frac{2}{x^2}(1+x), \quad R = x$$

Now, $$P + Qx = -\frac{2(1+x)}{x} + \frac{2(1+x)}{x} = 0$$

$\therefore$ $y = x$ is a part of C.F.

To find the solution of $\dfrac{d^2y}{dx^2} - \dfrac{2}{x}(1+x)\dfrac{dy}{dx} + \dfrac{2}{x^2}(1+x)y = 0$ ...(2)

Put $y = vx$ so that $\dfrac{dy}{dx} = x\dfrac{dv}{dx} + v$

and $$\frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$$

$\therefore$ Equation (2) becomes

$$x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} - \frac{2(1+x)}{x}\left[x\frac{dv}{dx} + v\right] + \frac{2(1+x)}{x^2}vx = 0$$

or $$x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} - 2(1+x)\frac{dv}{dx} - \frac{2(1+x)}{x}v + \frac{2(1+x)}{x}v = 0$$

or $$x\frac{d^2v}{dx^2} + 2(1-1-x)\frac{dv}{dx} = 0$$

or $$x\frac{d^2v}{dx^2} - 2x\frac{dv}{dx} = 0$$

or $$\frac{d^2v}{dx^2} = 2\frac{dv}{dx}$$

or $$\frac{dp}{dx} = 2p, \text{ where } p = \frac{dv}{dx}$$

or $$\frac{dp}{p} = 2dx$$

Integrating, $$\log p = 2x + \log c_0$$

or $$p = c_0 e^{2x}$$

i.e., $$\frac{dv}{dx} = c_0 e^{2x}$$

Integrating, $$v = \frac{c_0 e^{2x}}{2} + c_2 = c_1' e^{2x} + c_2'$$

Hence, the solution of (2) is

$$y = vx \quad i.e., \quad y = c_1' x e^{2x} + c_2' x$$

For finding the complete solution of (1), replace the arbitrary constants c_1' and c_2' by u, v which are functions of x

Let $y = u\,(x\,e^{2x}) + v\,(x)$ ...(3)

or $y = u\,.\,\phi_1(x) + v\,.\,\phi_2(x)$

where $\phi_1(x) = x\,e^{2x}$ and $\phi_2(x) = x$

$$W = \begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{vmatrix} = \begin{vmatrix} xe^{2x} & x \\ e^{2x}(2x+1) & 1 \end{vmatrix}$$

$$= xe^{2x} - xe^{2x}(2x+1)$$

$$= xe^{2x}[1 - 2x - 1] = -2x^2 e^{2x}$$

Now, $$u = -\int \frac{\phi_2(x) R}{W} dx + A = -\int \frac{x.x}{-2x^2 e^{2x}} dx + A$$

$$= \frac{1}{2}\int e^{-2x} dx + A$$

$$= \frac{1}{2}\left(\frac{e^{-2x}}{-2}\right) + A = -\frac{e^{-2x}}{4} + A$$

and $$v = \int \frac{\phi_1(x) R}{W} dx + B = \int \frac{xe^{2x}\,.\,(x)}{-2x^2 e^{2x}} dx + B$$

$$= -\int \frac{1}{2} dx + B = -\frac{x}{2} + B$$

From (3), $y = u\,(x\,e^{2x}) + vx$

or $$y = \left(-\frac{e^{-2x}}{4} + A\right) x\,e^{2x} + \left(-\frac{x}{2} + B\right) x$$

or $$y = A\,x\,e^{2x} + Bx - \frac{x}{4} - \frac{x^2}{2},$$

which is the complete solution of the given equation.

EXERCISE 6.4

Solve the following equations by the method of variation of parameters :

1. $\dfrac{d^2y}{dx^2} + y = x$

2. $\dfrac{d^2y}{dx^2} + 9y = \sec 3x$ [M.D.U 2013]

3. $\dfrac{d^2y}{dx^2} + a^2y = \text{cosec}\, ax$ [M.D.U 2008]

4. $\dfrac{d^2y}{dx^2} + y = \text{cosec}\, x$

5. $xy' - y = (x - 1)(y'' - x + 1)$

6. $\dfrac{d^2y}{dx^2} + a^2y = \sec ax$

7. $\dfrac{d^2y}{dx^2} + y = x \sin x$

8. $y'' - 6y' + 9y = \dfrac{e^{3x}}{x^2}$

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9. $(D^2 - 1)y = e^{-2x} \sin e^{-x}$ [M.D.U. 2016, 15; K.U. 2011]

10. $y'' - 2y' + 2y = e^x \tan x$ [M.D.U. 2011]

11. Verify that e^x and x are solutions of the homogeneous equation corresponding to $(1-x)y'' + xy' - y = 2(x-1)^2 e^{-x}$, $0 < x < 1$ and hence find its general solution.

12. Verify that $y = x$ and $y = x^2 - 1$ are linearly independent solutions of

$$(x^2+1)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0.$$

Find the general solution of $(x^2+1)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 6(x^2+1)^2$ [M.D.U. 2008]

13. Solve $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = x^2e^x$

14. Solve $\dfrac{d^2y}{dx^2} - y = \dfrac{2}{1+e^x}$ [K.U. 2008, 06, 03; M.D.U. 2012, 06, 05]

15. Solve $(x+2)\dfrac{d^2y}{dx^2} - (2x+5)\dfrac{dy}{dx} + 2y = (x+1)e^x$ [M.D.U 2013]

16. Solve $(1-x^2)\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} - (1+x^2)y = x$

17. Solve $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = x\log x$. [M.D.U. 2017]

ANSWERS

1. $y = c_1\cos x + c_2\sin x + x$

2. $y = c_1\cos 3x + c_2\sin 3x + \dfrac{1}{9}\cos 3x\log\cos 3x + \dfrac{x}{3}\sin 3x$

3. $y = c_1\cos ax + c_2\sin ax - \dfrac{x}{a}\cos ax + \dfrac{1}{a^2}\sin ax\log\sin ax$

4. $y = c_1\cos x + c_2\sin x - x\cos x + \sin x\log\sin x$

5. $y = c_1e^x + c_2x - (x^2 + x + 1)$

6. $y = c_1\cos ax + c_2\sin ax + \dfrac{1}{a^2}\cos ax\log\cos ax + \dfrac{1}{a}x\sin ax$

7. $y = c_1\cos x + c_2\sin x + \dfrac{x}{4}\sin x - \dfrac{x^2}{4}\cos x$

8. $y = (c_1 + c_2x)e^{3x} - e^{3x}\log x$

9. $y = c_1e^x + c_2e^{-x} - \sin e^{-x} - e^x\cos(e^{-x})$

10. $y = e^x[c_1\cos x + c_2\sin x] - e^x\cos x\log(\sec x + \tan x)$

11. $y = c_1x + c_2e^x + \left(\dfrac{1}{2} - x\right)e^{-x}$

12. $c_1(x^2-1) + c_2x + x^4 + 3x^2$

13. $y = c_1x + \dfrac{c_2}{x} + e^x - \dfrac{e^x}{x}$

14. $y = c_1e^x + c_2e^{-x} + e^x\log\dfrac{1+e^x}{e^x} - 1 - e^{-x}\log(1+e^x)$

15. $y = c_1(2x+5) + c_2e^{2x} - e^x$

16. $y = (c_1\cos x + c_2\sin x + x)\cdot\dfrac{1}{(1-x^2)}$.

17. $y = c_1x + c_2x^2 - x\left[\dfrac{(\log x)^2}{2} + \log x + 1\right]$